ASYMPTOTIC BEHAVIOR OF THE COUPLED KLEIN-GORDON-SCHRÖDINGER SYSTEMS ON COMPACT MANIFOLDS

César A. Bortot[1], Thales Maier Souza[2], Janaina Pedroso Zanchetta[1,2]

[1] *Federal University of Santa Catarina, Joinville, Brazil (c.bortot@ufsc.br)*
[2]*Federal University of Santa Catarina, Joinville, Brazil*
[1,2]*Paulista State University Júlio de Mesquita Filho, Ilha Solteira, Brazil*

Abstract: This work is concerned with a 2-dimensional Klein-Gordon- Schrödinger system subject to two types of locally distributed damping on a compact Riemannian manifold M without boundary. Making use of unique continuation property, the observability inequalities, and the smoothing effect due to Aloui, we obtain exponential stability results.

Keywords: Klein-Gordon-Schrödinger; Exponential stability; Compact manifolds; Differential equations on manifolds.

INTRODUCTION

The classical Klein-Gordon-Schrödinger system through Yukawa coupling, given by

$$\begin{cases} i\psi_t + \Delta\psi = \phi\psi \text{ in } \Omega \times (0,\infty) \\ \phi_{tt} - \Delta\phi + \mu^2\phi = |\psi|^2 \text{ in } \Omega \times (0,\infty) \\ \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1, \end{cases}$$

where, ψ is a complex scalar nucleon field while ϕ is a real scalar meson one and the positive constant μ represents the mass of a meson. Since it is considered a bounded domain with Dirichlet conditions, the term $\mu^2\phi$ does not affect the employed arguments in the proof of the asymptotic stability. In our case we will assume fields ψ and ϕ which has an average value of zero, thus the term $\mu^2\phi$ does not hinder the multiplier techniques employed in our results of decay rates. So, for simplicity, this term will be omitted. For more details on physical modeling see Yukawa (1955).

It is known, according to Tang and Ding (2008), that boththe nonlinear Klein–Gordon and Schrödinger equations with cubic interactions have wide applications in many physical fields such as nonlinear optics, nonlinear plasmas, condensed matter and so on. Besides, the system Klein-Gordon-Schrödinger may describe the dynamics of coupled electrostatic upper-hybrid and ion-cyclotron waves in auniform magnetoplasma.

In this paper, we consider the Cauchy problem of the following Klein-Gordon-Schrödinger equations through Yukawa interaction (P_j), $j = 1,2$.

$$\begin{cases} i\psi_t + \Delta\psi + i\alpha B_j(x,\psi) = \phi\psi X_\omega \text{ in } M \times (0,\infty) \\ \phi_{tt} - \Delta\phi + a(x)\phi_t = |\psi|^2 X_\omega \text{ in } M \times (0,\infty) \\ \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1 \text{ in } M \end{cases}$$

where (M, g) is a bidimensional compact Riemannian manifold without boundary and g represents your metric, αBj(x, ψ) is non-linear locally distributed damping, such that the constant α will be characterized later, ω is a region on M where the dissipative effect is effective, and χω is the characteristic function on ω. We study two types of damping, defined by:

$B_1(x,\psi) = b(x)(1-\Delta)^{1/2} b(x)\psi$ and
$B_2(x,\psi) = b(x)(|\psi|^2 + 1)\psi$.

We assume that a(·), b(·) are non-negative functions satisfying

a, b ∈ $W^{1,\infty}(M) \cap C^\infty(M)$

$a(x) \geq a_0 > 0$ in ω, and $b(x) \geq b_0 > 0$ in ω, where ω is an open subset of M such that $meas(\omega) > 0$ satisfying geometric control condition.

Remark 1. Observe that, the couple (ω, T_0) satisfies the geometric control condition (GCC, in short) if every geodesic of M, traveling with speed 1 and issued at $t = 0$ enters the open set ω before the time T_0.

We are interested in uniform decay results for Klein-Gordon-Schrödinger equations with the damping effect. The Klein-Gordon-Schrödinger system with the dissipative mechanisms was considered extensively in the literature, for instance, the exponential decay of Klein-Gordon-Schrödinger system with full damping in both equations holds. These results due to Cavalcanti and Domingos

Cavalcanti (2000) they have used a perturbed energy method to guarantee exponential decay rates. On the other hand, a uniform decay result holds, considering locally distributed damping into the wave equation and full damping into the Schrödinger equation. This was proven by the Bisognin et al. (2008).

An interesting result of exponential decay considering Klein-Gordon-Schrödinger system with localized damping in both equations are due the Almeida et al. (2018).

MATERIAL AND METHODS

In order to obtain the desired results, the tools used in the works already mentioned above were crucial.

The main features of this work are as follows:

(i) We consider the Klein-Gordon-Schrödinger system with localized damping and provide a regular function $f : M \to \mathbb{R}$ that allows us to apply the multiplier method and Unique continuation property.

(ii) We establish uniform decay rates of both systems. More specifically, we use multiplier functional combined with the regularizing effect due to Aloui on manifolds. In aditional, by using geometric control conditions on ω, we obtain observability inequalities in the linear wave and the Schrödinger equation.

It is worth mentioning that the results of unique continuation are closely related to the existence of this multiplier. On the other hand, the effective dissipation region ω needs the properties of this mentioned multiplier. The general idea is construct a function $f : M \to \mathbb{R}$ and define an open subset V ⊂ M where $meas(\mathrm{V}) \geq meas(\mathrm{M}) - \epsilon$ for every $\epsilon > 0$, such that f satisfies conditions related to it is Hessian, gradient and Laplacian in V. Thus, we can define ω (see Figure 1) as the open subset such that,

$\omega \supset\supset (\mathrm{M} \backslash \mathrm{V})$.

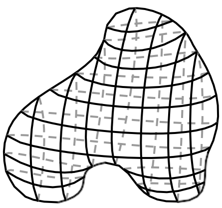

Figure 1. Blank, we have an arbitrarily large area, free of dissipative effects while in black, it is in the region ω where the dissipative effect is effective, this region can be chosen arbitrarily small, both totally distributed on M.

RESULTS

We shall use standard Sobolev spaces on Riemannian manifolds. We define

$$V := \{y \in H^1(\mathrm{M});\ \int_M y(x)\, d\mathrm{M} = 0\}$$

$$H := \{y \in L^2(\mathrm{M});\ \int_M y(x)\, d\mathrm{M} = 0\}$$

The energy associated to problems (P_j), $j = 1,2$, is defined by

$$E(t) = \frac{1}{2}\int_{\mathrm{M}} |\psi|^2 + |\nabla\phi|^2 + |\phi_t|^2\, d\mathrm{M}. \qquad (1)$$

Theorem 1. Assume that $5(2a_0b_0)^{-1} \leq \alpha$ in the problem (P_2). Then, given $(\psi_0, \phi_0, \phi_1) \in \{V \cap \mathrm{H}^2(\mathrm{M})\}^2 \times V$ problems *(P1)* and *(P2)* has a unique regular solution satisfying

$$\psi \in L^\infty\left(0, \infty;\ V \cap H^2(\mathrm{M})\right),$$
$$\psi_t \in L^\infty\left(0, \infty;\ L^2(\mathrm{M})\right),$$
$$\phi \in L^\infty\left(0, \infty;\ V \cap H^2(\mathrm{M})\right)$$
$$\phi_t \in L^\infty\ (0, \infty;\ V) \text{ and}$$
$$\phi_{tt} \in L^\infty\left(0, \infty;\ L^2(\mathrm{M})\right).$$

Proof. See Bortot et al. (2022).

Theorem 2. Suppose that the hypotheses of Theorem 1 holds. In addition, $\alpha > \frac{a_0^{-1} b_0^{-4} d}{2}$ or d is sufficiently small. Then, there exist C, γ positive constants such that following decay rate holds

$E(t) \leq Ce^{-\gamma t}E(0),\ for\ all\ t \geq 0,$

for every regular solution of problem *(P1)* satisfying the Theorem 1, provided the initial data are taken in bounded sets of *H*.

Proof. See Bortot et al. (2022).

Theorem 3. Suppose that the hypotheses of Theorem 1 holds. We assume that ω is an open subset of M such that $meas(\omega) > 0$ and satisfying the geometric control condition. Then, there exist C, γ positive constants such that following decay rate holds

$E(t) \leq Ce^{-\gamma t}E(0),\ for\ all\ t \geq 0,$

for every regular solution of problem *(P2)* satisfying the Theorem 1, provided the initial data are taken in bounded sets of *H*.

Proof. See Bortot et al. (2022).

Remark 2. Observe that assumption $5(2a_0b_0)^{-1} \leq \alpha$ is necessary for the well-posedness of problem *(P2)*. However, this is not a drawback in the method since the constants a_0 and b_0 are used just to localize the dissipative effect. On the other hand, to take α sufficiently large is natural to guarantee the dissipativity of the system. The second one has been considered previously and surely it is much more natural.

CONCLUSION

We can conclude that the main objectives related to the exponential stability of the energy associated with the Klein-Gordon-Schrödinger coupled system were achieved. Future objectives related to these problems would be to further improve the region where energy dissipation has to be effective. For this, we believe that specific geometric characteristics of certain Riemannian manifolds will be essential.

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