CONFIABILIDADE EM ANÁLISE LIMITE INCERTA DE UM PÓRTICO

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RESUMO: *A incerteza é parte indissociável dos parâmetros envolvidos na análise de confiabilidade de estruturas. Deste modo, um conjunto selecionado de parâmetros assume um comportamento não determinístico neste tipo de análise. A referida variabilidade nos parâmetros estruturais muitas vezes remete a uma análise de confiabilidade. Portanto, neste artigo, uma análise de confiabilidade estrutural de um pórtico é realizada sob análise limite no programa Matlab®. Utilizando a simulação de Monte Carlo para propagar as incertezas, as cargas aplicadas e o limite de escoamento do material apresentam comportamento probabilístico. O objetivo principal é obter a probabilidade de falha do pórtico, sua carga de colapso e o fator de colapso relacionado ao colapso plástico nas condições descritas. Adicionalmente, o modelo probabilístico é comparado ao determinístico no contexto do limite plástico. Diante dos resultados, a abordagem determinística não resultou em nenhuma falha do sistema, enquanto a abordagem probabilística apresentou uma probabilidade de falha máxima de 0,05%. Portanto, em sistemas estruturais em que o fator de segurança plástico é relativamente pequeno, uma análise determinística pode incorrer num risco mais provável de falha não prevista do sistema.*

PALAVRAS-CHAVE: *Confiabilidade, Incerteza, Análise limite, Estruturas.*

RELIABILITY IN UNCERTAIN LIMIT ANALYSIS OF A FRAME

ABSTRACT: *Uncertainty is an inseparable part of the parameters involved in the reliability analysis of structures. Given this, a selected set of parameters assumes a non-deterministic behavior in this type of analysis. The referred variability in structural parameters often addresses a reliability analysis. Therefore, in this paper, a structural reliability analysis of a frame is conducted within the limit analysis in the Matlab® program. Using Monte Carlo simulation to propagate the uncertainties, the applied loads, and the material yield strength present probabilistic behavior. The main objective is to obtain the failure probability of the frame, its collapse load, and the collapse factor related to the plastic collapse under the conditions described herein. Moreover, the probabilistic model is compared to the deterministic one in the context of the plastic limit. Given the results, the deterministic approach yielded no system failure, whereas the probabilistic approach presented a maximum failure probability of 0.05%. Therefore, in structural systems in which the plastic safety factor is relatively small, a deterministic analysis may incur a more probable risk of non-predicted system failure.*

KEYWORDS: *Reliability, Uncertainty, Limit analysis, Structures.*

1. INTRODUCTION

In the context of a mechanical design, it is common to assume that the involved quantities, such as geometric, mechanical properties, loading, support localizations, conditions, etc. are exact, constant, and deterministic. Simplification, requirements, and regulations are some of the plausible justifications for conducting a deterministic structural analysis. Although these justifications are adequately suited for some situations, a more realistic mathematical modeling should include variable parameters because, even if the established design and production controls are at a high threshold, the variation of involved parameters is intrinsic to the design and production systems. Given this, the addressed variation corresponds to a set of probable values, instead of a single deterministic value.

1.1 Limit analysis

There are two theorems on which limit analysis is based (a) static or lower bound (Gvozdev, 1938); and (b) kinematic or upper bound (Drucker *et al.*, 1952).

The fundamentals of limit analysis theory are presented in detail in (Fuschi, 2015), (Chen and Han, 1988), (Maugin, 1992), and (Lubliner, 2008), among others. The aim is to find the monotone externally applied collapse load of a structure. In the case of statically indeterminate structures subjected to the plastic criteria, the collapse load is always higher than the load to induce the first point, line, or region of yielding. When the required number of plastic hinges (mechanisms) takes place, the collapse then occurs. Thenceforth, the structural system loses its static indeterminacy condition. Therefore, the mathematical meaning of the collapse factor addresses how much bigger the collapse load is related to the applied load magnitude.

Limit analysis was developed as an alternative to the incremental elastic-plastic analysis. Due to its more promptness compared to the incremental analysis, it is denominated a direct method (Lubliner, 2008). Therefore, limit analysis is widely applied in situations in which the load history is not relevant because it deals with time-independent loads. Thus, the limit analysis may significantly reduce the time to obtain the limit load of structural systems subjected to the plastic collapse criteria. From this perspective, limit analysis provides simplification to the calculation when the main objective is to find the collapse load.

Due to its mathematical features, limit analysis can be implemented via algorithms. For example, (Kaveh et al., 2008) reported the limit analysis of frames via heuristic algorithms and introduced the so-called ant colony system (ACS). (Hofmeyer and Delgado, 2012) compared the performance of genetic algorithms and co-evolutionary ones in the optimal design of structures.

(Jahanshah *et al*., 2017) compared ant colony algorithms, genetic algorithms, and neural networks in what refers to the collapse load of planar frames. (Greco *et al*., 2019) proposed a modified ant colony algorithm, which reduced the computational effort to obtain the optimal solution of planar frames. (Cacho-Pérez, 2024) obtained the collapse mechanism and the collapse factor for frames with slender structural members of variable cross-section and arbitrarily distributed loads.

As mentioned, there are two theorems on which limit analysis is based. The kinematic (or upper bound) theorem is the selected one to develop the calculations in this paper. The term "upper bound" refers to the fact that the limit load is greater than or equal to the true value of the collapse factor, according to (Drucker *et al.*, 1952).

Structural failure occurs when the external work on the structure exceeds the internal work (which the structure can withstand). Therefore, let W_E be the external work on the structure, W_I the internal work, F_i the i-th internal force, P_i the externally applied load, u_i the i-th geometrically possible displacement field, n the number of internal forces, V the volume domain, and σ the stress field. Thenceforth, Equation 1 expresses the failure condition:

$$W_E = \sum_{i=1}^{n} P_i\, u_i > W_I = \int_V W_I\left(F_1, F_2, \ldots, F_i, \ldots, F_n\right) dV = \int_V \sigma \cdot dV\,. \tag{1}$$

The upper bound theorem states that the internal work must be compared to the external work. The ratio between the internal work and the external work is the definition of the upper bound collapse factor, α_U, which is greater than or equal to the real collapse factor (α_R). This condition is expressed in Equation 2:

$$\alpha_U = \frac{\int_V W_I\left(F_1, F_2, \ldots, F_i, \ldots, F_n\right) dV}{\sum_{i=1}^{n} P_i\, u_i} \geq \alpha_R. \tag{2}$$

1.2 Uncertainty

According to (Roy and Oberkampf, 2011), uncertainty can be modeled as probabilistic (aleatory-type) and/or non-probabilistic (epistemic-type). The former is represented by a probability density function (PDF) and/or a cumulative density function (CDF). This type of uncertainty cannot be eliminated due to its intrinsic variability. (Nonato, 2023) conducted an uncertain probabilistic fatigue analysis with all the uncertain input quantities as probabilistic ones. The epistemic-type uncertainty is represented by an interval when there is a lack of information about the parameter. (Kumar and

Balu, 2023) proposed a modified analysis of structural systems under epistemic uncertainty. A hybrid analysis (that combines both aleatory and epistemic types) was performed in the context of a truss in a fatigue limit state (Nonato, 2020), for example.

Therefore, the phenomenological uncertainties can be mathematically modeled in a structural reliability framework. (Tran and Staat, 2014) state that to perform this reliability evaluation, adequate mechanical models and analysis procedures are required for the process of modeling accurate limit states. Moreover, the aleatory parameters have to be suited to obtain a meaningful reliability assessment.

1.3 First-order reliability method (FORM)

FORM aims at approximating the limit state function by a linear surface at the design point in the design space, which defines it as an integration method of analytical probability, according to (Tran and Staat, 2014). The output is the failure probability of bodies under loading. The gradient of the approximation function must be known and the probabilistic variables be continuous. The origin of FORM comes from the second-moment methods, thus denominated due to the first and second moments of the probabilistic variables, as stated by (Haldar and Mahadevan, 2000). Therefore, FORM includes two approaches: first-order second-moment (FOSM) and advanced first-order second-moment (AFOSM). The difference between the former and the latter is that the information about the distribution of random variables is ignored in the former but considered in the latter.

FORM is based on a first-order approximation of Taylor's series of the performance function linearized at the mean values of the involved probabilistic variables and applies second-moment statistics of the referred variables. Let the collapse load be denominated by the Gaussian variable F_C and the applied load be the Gaussian variable F_A, being both statistically independent. The variable Z, which is also Gaussian, represents the result of the limit state function ($Z = F_C - F_A$). Given this, the failure can be represented by $F_C < F_A$ or $Z < 0$. Equation 3 represents the limit state function, where $\mathbf{X}$ is the vector of all possible realizations:

$$Z(\mathbf{X}) = F_C(\mathbf{X}) - F_A(\mathbf{X}), \quad (3)$$

where the structure is safe if the result is positive; the null result corresponds to the achievement of the limit state, and the negative result represents the failure (collapse) status.

For the Gaussian distribution, the failure probability can be expressed in terms of the relation between the mean (μ_Z), and the standard deviation (σ_Z) of the limit state variable Z. This relation can be represented in terms of the relation between the difference of means μ_{F_C} and μ_{F_A}, and the Euclidean norm of standard deviations λ_{F_C} and λ_{F_A}. This relation is the so-called reliability index (Equation 4):

$$\beta = \frac{\mu_Z}{\lambda_Z} = \frac{\mu_{F_C} - \mu_{F_A}}{\sqrt{\lambda_{F_C}^2 + \lambda_{F_A}^2}}. \tag{4}$$

The reliability index is the parameter with which the failure probability is calculated via the cumulative standard normal distribution function, as per Equation 5:

$$P_F = \Phi(-\beta) = 1 - \Phi(\beta). \tag{5}$$

1.4 Work scope

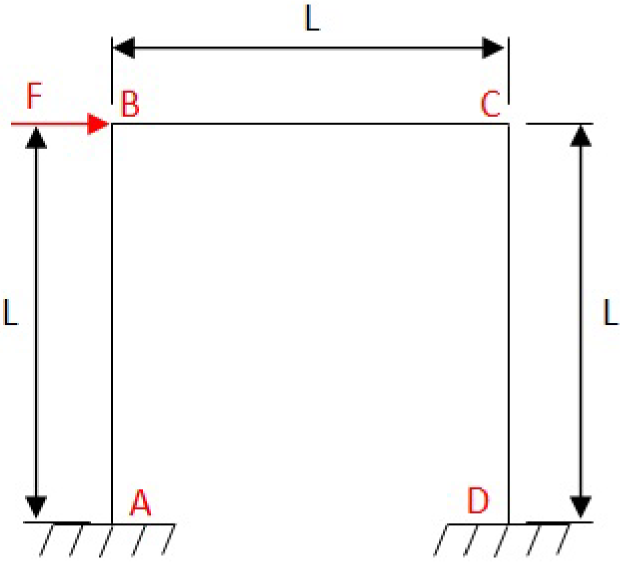
The reliability analysis of a frame is conducted by considering the plastic limit criteria of the material via limit analysis. The reliability index, collapse factor, limit load, failure probability, and plastic safety factor of the structure are the main output data obtained in this paper. In addition, the comparison of the results from deterministic and probabilistic approaches is made in the context of this reliability analysis. The externally applied loads and the material yield strength are uncertain, thus assuming a probabilistic behavior. The comparison parameters between deterministic and probabilistic approaches are the deterministic plastic safety factor and failure probability, respectively.

2. MATERIALS AND METHODS

The reliability analysis conducted herein is based on the adoption of the elastic-perfectly material model. In other words, there is no strain hardening in the plastic zone of the material, i.e. the yield function is not allowed to assume values greater than zero. Moreover, the material properties are assumed to be isotropic and homogeneous, all the structural elements are constituted of the same material (ASTM A36), and the non-probabilistic quantities are considered deterministic.

The frame example calculated in this paper is presented in Figure 1, in which a load F is applied at node B in the horizontal from left to right. In this context, the applied load, F_A, is equal to F. The lengths of all the structural elements are the same (L), nodes B and C are free to displace, and nodes A and D are fixed. The cross-section of the structural elements is rectangular with base b and height h, oriented to provide the maximum moment of inertia. The deterministic quantities are presented in Table 1.

FIGURE 1. The frame of four nodes subjected to a load $F = F_A$ at B.



Source: Author (2024).

TABLE 1. Deterministic quantities in the frame example.

Base of the rectangle, b, (mm)	Height of the rectangle, h, (mm)	Length of the element, L, (mm)	Number of realizations
30	40	350	1×10^6

Source: Author (2024).

The uncertain parameters are Gaussian, in which their means and standard deviations are the available information. There are two probabilistic variables: (a) material yield strength (S_y); and (b) the applied load (F_A). Even though it often requires larger sample sizes, the selected uncertainty propagation method is Monte Carlo sampling because it is easier to implement than Latin hypercube sampling or polynomial chaos expansion, for example. The uncertain parameters and their main characteristics are presented in Table 2.

TABLE 2. Uncertain input parameters in the frame example.

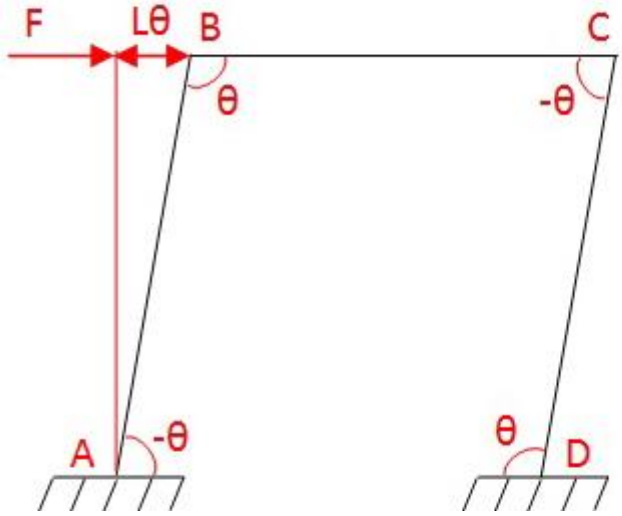
Uncertain parameter	Mean	Standard deviation
Yield strength, S_y, MPa	250.00	5.00
Applied load, $F = F_A$, kN	31.00	0.62

Source: Author (2024).

3. RESULTS

As part of the results, the possible collapse configuration considered in the reliability analysis of the frame is presented in Figure 2, in which four plastic mechanisms are simultaneously formed. Previously to the application of the load, there were four straight angles θ between the structural members. During the application of the load, two of them assumed an obtuse angle while the other two turned to be acute.

FIGURE 2. The kinematic failure mechanism of the frame.



Source: Author (2024).

Given the failure mechanism presented in Figure 2, now the upper bound theorem can be applied to solve for the collapse factor (α), and, therefore, to the collapse load (F_C). Equation 6 expresses the collapse load (F_C) for the frame case, where M_P stands for the plastic moment of the cross-section,

$$F_C = \frac{4\,M_P}{L}. \tag{6}$$

Equation 7 expresses the plastic moment for a rectangular cross-section bar:

$$M_P = \frac{b\,h^2\,S_y}{4}. \tag{7}$$

Thus, substituting Equation 7 into Equation 6, the collapse load (F_C) can now be represented by Equation 8:

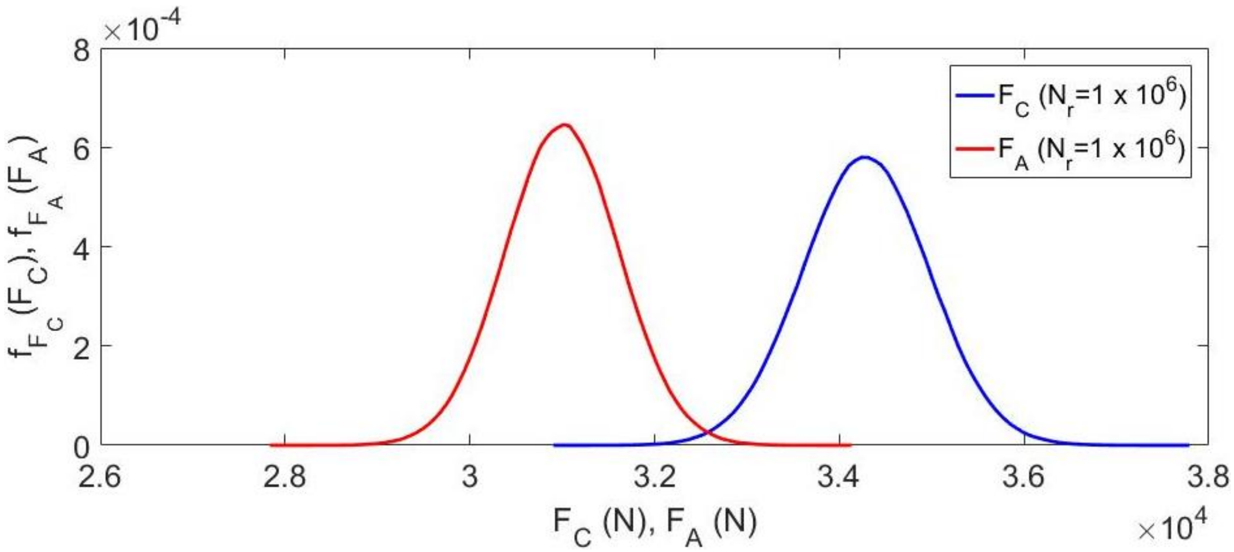
$$F_C = \frac{b\,h^2\,S_y}{F_A\,L}. \tag{8}$$

Therefore, the amplification factor is obtained via Equation 9:

$$\alpha = \frac{F_C}{F_A}. \tag{9}$$

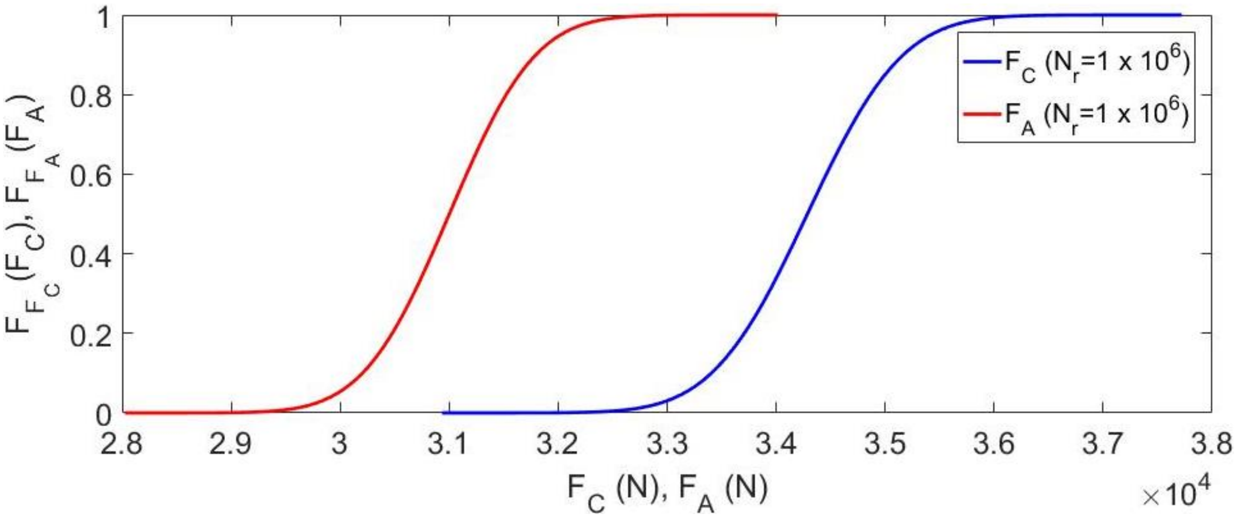
The probability distribution functions (PDFs) and the cumulative distribution functions (CDFs) of the collapse and applied loads related to the frame are presented, in Figure 3 and Figure 4, respectively.

FIGURE 3. PDFs of applied load $f(F_A)$ and collapse load $f(F_C)$ for the frame.



Source: Author (2024).

FIGURE 4. CDFs $f(F_A)$ and $f(F_C)$ of applied and collapse loads of the frame, respectively.



Source: Author (2024).

Table 3 presents the results of the mean, standard deviation, minimum value, and maximum value for the plastic moment, collapse load, collapse factor, reliability index, and failure probability.

TABLE 3. Results for the main uncertain output parameters.

Uncertain parameter	Mean	Standard deviation	Minimum value	Maximum value
Plastic moment, $M_P(N.mm)$	3.00 x 10^6	0.06 x 10^6	2.67 x 10^6	3.29 x 10^6
Collapse load, $F_C(N)$	34290	690	30560	37590
Collapse factor, α	1.11	0.03	0.96	1.27
Reliability index, β	3.55	0.02	0.96	1.26
Failure probability, $P_F(\%)$	193.89 x 10^{-4}	499.82 x 10^{-4}	1.40 x 10^{-4}	519.92 x 10^{-4}

Source: Author (2024).

If the frame was solved exclusively via the deterministic approach, the system would be safe because the mean applied load ($31.00\,kN$) is lower than the mean of the limit load ($34.29\,kN$), thus yielding a plastic safety factor of 1.11 (a safety margin of 11%). Conversely, when the probabilistic approach is applied, the system is safe in approximately 99.95% of the realizations. Then, if the uncertainties were ignored, or not considered in the calculation, the structural system would be safe. However, if the applied load and the material yield strength are probabilistic, which is possible in mathematical modeling to reflect the process uncertainties, the structural system will present a non-null failure probability.

4. CONCLUSIONS

The structural reliability analysis of a planar frame was performed considering the plastic criteria via limit analysis, where the applied loads and material yield strength have a probabilistic behavior. Thenceforth, the following conclusions can be extracted from this paper:

(1) Plastic *vs*. elastic analysis: under the same circumstances, plastic analysis addresses collapse loads higher than those in elastic analysis. However, the deformation magnitudes in plastic analysis are higher than those in elastic ones. It is often applied when it is possible to maximize the load-carrying capacity. Given this, the plastic analysis produces lighter structural systems.

(2) Incremental plastic analysis *vs*. limit analysis: the latter method is faster in cases where the collapse load has to be obtained without worrying about the loading history. Moreover, as limit analysis is an optimization problem it becomes relatively easy to code it into a program.

(3) Deterministic *vs*. probabilistic analysis: in the frame problem, the non-null failure probability from the reliability analysis was compared to the null failure condition from the deterministic analysis. In cases of low safety margin, deterministic analysis is more prone to a sudden non-predicted system failure in a more realistic modeling of the phenomena.

(4) Moreover, to achieve a more realistic modeling of the situation, often uncertainty modeling and calculation should be conducted. Mathematical models closer to the real situations to which they correspond are possibly more valuable in decision-making processes.

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RESPONSIBILITY DECLARATION

The author is the only one responsible for this work.

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