Statistical Inference for Generalized Power-Law Process in Repairable Systems

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Resumo

Repairable systems are often used to model the reliability of restored components after a failure is observed. Among various reliability growth models, the power law process (PLP) or Weibull process has been widely used in industrial problems and applications. In this article, we propose a new class of model called the generalized PLP (GPLP), based on change points. These can be treated as known or unknown parameters, or interpreted as failure times. Herein, we consider the impact of all or some fixes on the failure intensity function. In this context, unlike the usual PLP, the GPLP is not restricted to the assumption of minimal repair (MR). Other situations, such as perfect, efficient, and harmful repair, can be considered. We present some special cases of the GPLP, such as the main models used to analyze repairable systems under the assumption of imperfect repair. The estimators of the proposed model parameters were obtained using the maximum likelihood method. We evaluated the performance of the parameter estimators through Monte Carlo (MC) simulations. The proposed approach is fully illustrated using two real failure time datasets.

Palavras-chave: Generalized Power Law Process; Non- Homogeneous Poisson Process; repairable systems, Point of change, Imperfect Repair

1 Introduction

The PLP Power Law Process, also known as the Weibull process, is a famous particular case of the Non-Homogeneous Poisson Process (NHPP) proposed by Crow (1974), often used in the modeling of failure times of repairable systems. An assumption for using the PLP intensity function is that, after repair, the system returns to the stage immediately before the failure, that is, as bad as old (ABAO - *as bad as old*). This type of repair is known in the literature as RM. This assumption is not valid for all situations, there may be cases where the systems became better than before the failure (or even as good as new) or deteriorated after repair and, therefore, it is not suitable to use the PLP for model the failure times of these systems. Showing a limitation of the PLP.
Many models have been proposed aiming to capture this repair effect on the failure intensity function. Brown and Proschan (1983) introduced the concept of imperfect repair (IR),

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where the system is as good as new with probability p and as bad as before with probability $1-p$. Kijima et al. (1988) proposed the generalized renewal process through the virtual age mechanism. According to Lindqvist (2006), the type of repair (minimum or perfect) must be informed every repair action. In real applications, however, type information is rarely available. Doyen and Gaudoin (2004) introduced two classes of imperfect repair models, they are: (i) ARA - *Arithimetic Reduction of Age*; and (ii) ARI -*Arihtimetic Reduction of Intensity*, for which the repair effect is characterized by a parameter that induces a change in the intensity of the failure, unlike the Kijima et al. (1988) model where it is necessary to observe the type of repair.
In this work we propose a generalized PLP (GPLP) that allows interpreting repair times as change points, considering the impact of some repairs in the process or also treating change points with jumps or drops and quantify their effect on the failure intensity function . GPLP unlike the usual PLP is not restricted to the MR assumption, it is possible to consider other situations like perfect, efficient and harmful repair. The GPLP is proposed with the aim of overcoming limitations in the modeling of imperfect repairs, conditioning the failure intensity function to the system situation to maintain the decreasing direction of the repair effect on the intensity function, interpreting β as a quantifier of the state intensity of the system, as well as an indicator of the situation of the system, whether it is in a state of degradation or renovation. The effect of repair acts to reduce the age of the system.

2 The Generalized PLP (GPLP)

The Generalized Power Law Process (GPLP) is a Non-Homogeneous Poisson Process with failure intensity function defined as

$$\lambda_{GPLP}(t) = \sum_{i=1}^{p+1} \frac{\mu_T \beta_{i-1}}{T + g(\theta_{i-1})\tau_{i-1}} \left[t - \text{sign}(\beta_{i-1} - 1)h(\theta_{i-1})\tau_{i-1}\right]^{\beta_{i-1}-1} \text{I}(\tau_{i-1} < t \leq \tau_i), \quad (1)$$

where p is the number of changes in the process, $0 = \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} = T$, $T > 0$, $\mu_T > 0$, $\beta_{i-1} > 0$, $\theta_{i-1} \in \mathbb{R}$, $\text{sign}(\beta_{i-1} - 1) = 1$, if $\beta_{i-1} \geq 1$ and $\text{sign}(\beta_{i-1} - 1) =$ -1, if $\beta_i < 1$ and $\text{I}(\tau_{i-1} < t \leq \tau_i)$ is an indicator function, for $i = 1, \ldots, p+1$. If $t \leq 0$, we will consider $\lambda_{GPLP}(t) = 0$. The functions $g(\theta_{i-1})$ and $h(\theta_{i-1})$ must satisfy the conditions:

1. are continuous and not decreasing in θ_{i-1};
2. for $\beta_{i-1} \leq 1$, $h(\theta_{i-1}) \geq -1$ and for $\beta_{i-1} \geq 1$, $h(\theta_{i-1}) \leq 1$;
3. $g(\theta_{i-1}) \geq -1$, $g(0) = 0$ and $h(0) = 0$.

Consider the case where θ_{i-1} has a parametric space in $[-1, 1]$ and when $h(\theta_{i-1})$ is not explicit, let's define the function $h(\theta_{i-1})$, strictly increasing, like:
1) For $\beta_{i-1} \geq 1$, $h(\theta_{i-1}) : (-1, 1] \to (-\infty, 1]$, such that

$$h(\theta_{i-1}) = \begin{cases} \log\left(\dfrac{1+\theta_{i-1}}{1-\theta_{i-1}}\right) 10^k, & \text{if } -1 < \theta_{i-1} \leq 0, \\ \theta_{i-1}, & \text{if } 0 \leq \theta_{i-1} \leq 1, \end{cases}$$

2) For $\beta_{i-1} < 1$, $h(\theta_{i-1}) : [-1, 1) \to [-1, \infty)$, such that

$$h(\theta_{i-1}) = \begin{cases} \theta_{i-1}, & \text{if } -1 \leq \theta_{i-1} \leq 0, \\ \log\left(\dfrac{1+\theta_{i-1}}{1-\theta_{i-1}}\right)10^k, & \text{if } 0 \leq \theta_{i-1} < 1, \end{cases}$$

where $k = 1,\ 2, \ldots$ is a constant.
Finally, when the function $g(\theta_{i-1})$ is not explicit, we will take $g(\theta_{i-1}) = \theta_{i-1}$. However, $g(\theta_{i-1})$ and $h(\theta_{i-1})$ can be defined in other ways, for example, we can take $g(\theta_{i-1})$ like

$$g(\theta_{i-1}) = \begin{cases} \dfrac{1 - e^{|\theta_{i-1}|}}{e-1}, & \text{if } -1 \leq \theta_{i-1} \leq 0, \\ \dfrac{e^{\theta_{i-1}} - 1}{e-1}, & \text{if } 0 \leq \theta_{i-1} \leq 1, \end{cases}$$

which implies smoother effects of θ_{i-1} on $g(\theta_{i-1})$. Also, the functions $g(\theta_{i-1})$ and $h(\theta_{i-1})$ can be null, that is, $g(\theta_{i-1}) = 0$ and $h(\theta_{i-1}) = 0$ for all θ_{i-1}.
Through the GPLP it is possible to derive several models for analysis of repairable systems, next, we will obtain their particular cases, considering the change points as unknown, known parameters or considering as observed times at which system failures occurred. In this way, GPLP can be seen as a class of parametric models for the analysis of repairable systems.

3 GPLP for repairable systems

Now, considering that the change times are the observed times when the failures occurred, that is, $\tau_i = t_i$, for all $i = 1 \ldots, p$ and $p = n$, where n is the number of failures up to T and t_i is the observed time of the ith failure, so in this situation, τ_i is not interpreted as a parameter but as an observable value of failure times.

- **Homogeneous Poisson Process (HPP)**

The simplest case of GPLP, to obtain the HPP we do $\theta = 0$ and $\beta = 1$, therefore:

$$\lambda_C(t) = \frac{\mu_T}{T}\mathrm{I}(0 < t \leq T),$$

indicating that the instantaneous rate of failure is constant over time t.

- **Minimum Repair Model (RM)**

The GPLP parameterization to obtain the model for repairable systems with minimal repair (RM), that is, there is no reduction in the intensity of failure caused by the effect of repair θ, therefore $\theta = 0$, we therefore obtain:

$$\begin{aligned} \lambda_{RM}(t) &= \sum_{i=1}^{n+1} \frac{\mu_T}{T}\beta t^{\beta-1}\mathrm{I}\ (t_{i-1} < t \leq t_i), \\ &= \frac{\mu_T}{T}\beta t^{\beta-1}\mathrm{I}(0 < t \leq T) \end{aligned}$$

where $\beta \geq 1$, $t_0 = 0$, $t_{n+1} = T$ and $\sum_{i=1}^{n+1} \mathrm{I}(t_{i-1} < t \leq t_i) = 1 = \mathrm{I}(0 < t \leq T)$. Note that the intensity function of the RM model is the PLP intensity ($\lambda_{PLP}(t)$), this means that if the effect $\theta_i = 0$ at all GPLP change points, then it is equivalent to having no change points in terms of intensity function, that is, $p = 0$, since all effects are null.

- **Perfect Repair Model (RP)**

For $\theta = 1$ and $g(\theta) = 0$ in the GPLP, we obtain the model for repairable systems with perfect repair (RP), like this:

$$\lambda_{RP}(t|H_{t-}) = \sum_{i=1}^{n+1} \frac{\mu_T}{T} \beta \left(t - t_{i-1}\right)^{\beta-1} \mathrm{I}(t_{i-1} < t \leq t_i),$$

where $\beta \geq 1$, $t_0 = 0$, $t_{n+1} = T$ and H_{t-} is the history of failures immediately prior to t.

- **Imperfect Repair Model ($\mathbf{ARA_1}$)**

The GPLP parameterization to obtain the ARA_1 model is given with $0 < \theta < 1$ and we define $g(\theta) = 0$ for all θ, and thus we obtain:

$$\lambda_{ARA_1}(t|H_{t-}) = \sum_{i=1}^{n+1} \frac{\mu_T}{T} \beta \left(t - \theta t_{i-1}\right)^{\beta-1} \mathrm{I}(t_{i-1} < t \leq t_i),$$

where $\beta \geq 1$, $t_0 = 0$, $t_{n+1} = T$.

4 Application

Consider failure data for a steam hood, with operating time in number of days (Syamsundar et al. (2021) and Syamsundar and Naikan (2008)). In the research, the authors Syamsundar et al. (2021) identified that the best model for the data set is the XARA_{15}-GRA with linear logarithmic baseline ($\log \hat{L} = -64.4$), however the best model with PLP as baseline was XARA_{∞} - GRA - PLP with $\log \hat{L} = -66.5$. Syamsundar and Naikan (2008) concluded that the HPP segmented model with two change points (HPP MCP, $\log \hat{L} = -64.211$), $\hat{\tau}_1 = 128$ and $\hat{\tau}_2 = 244$ days, fits the dataset better compared to HPP models without changepoints ($\log \hat{L} = -70,733$), NHPP ($\log \hat{L} = -70,665$) and ARI_1 ($\log \hat{L} = -70,662$). The data set consists of 17 records of failure times, over a period of just over a year, as shown in Table 1.

Table 1: Set of failure times of a steam hood in days.

27	92	128	133	140	151	153	158	169
28	183	191	212	234	244	300	401	

Table 2: Parameters of the model RPI(p) used to model the steam hood failure times.

n	p	$\hat{\beta}$ (CI 95%)	$\hat{\mu}_T$ (CI 95%)	$\hat{\theta}$	τ_1	τ_2	$\log \hat{L}$	AIC
17	2	2.0627 (1.2751, 2.8500)	0.0723 (0, 0.3790)	1	244	300	-63.75	133.50

5 Conclusion

In this paper, we propose a new NHPP that generalizes the Power Law Process, based on change points, named as GPLP. We emphasize that the change points can be interpreted as known and unknown parameters and as observable failure times. From the GPLP, it is possible to obtain, as particular cases, the main models for repairable systems, such as the PLP itself, several models based on change points, perfect and imperfect repair models such as ARA1 and ARAM1, as well new models based on all or part of the failure times are added, which we call Completely Imperfect Repair Models - CIR and Partially Imperfect Repair models with change points - PIR(), respectively. The GPLP allows the intensity function not to remain parallel after the repairs. This new alternative for modeling repairable systems with GPLP expands the applications in real data of failure times.

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