EXPERIMENTAL AND NUMERICAL STUDY OF TRANSIENT COOLING

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Abstract: Transient heat transfer in a cooling water sample is modelled using Newton's law of cooling. Experimental temperature data are compared with the analytical solution and with forward Euler and fourth-order Runge–Kutta (RK4) approximations. Both methods are accurate at small time steps, but Euler deviates strongly as the step grows, while RK4 stays close to the analytical and experimental results, highlighting the role of method choice and discretization.

Keywords: Numerical methods; Newton's law of cooling; Runge–Kutta method; Euler method; transient heat transfer

INTRODUCTION

Heat transfer and temperature evolution play a fundamental role in many engineering applications, including chemical processing, thermal management, and energy systems. In such systems, transient temperature behaviour is commonly described using differential equations that represent the balance between energy storage and heat exchange. One of the simplest and most widely used models is Newton's law of cooling, which states that the rate of temperature change is proportional to the temperature difference between an object and its surroundings (Moran et al., 2014).

While analytical solutions for first-order differential equations are well established, real-world systems often involve complexities such as nonlinearities and coupled transport phenomena that prevent closed-form solutions. In these cases, numerical methods provide a practical alternative by approximating the solution through discrete time stepping (Gilat and Subramaniam, 2014). Methods such as the forward Euler scheme and higher-order Runge–Kutta methods are widely applied in engineering simulations, including reactor modelling and thermal system design (Gilat and Subramaniam, 2014).

Beyond theoretical convenience, numerical methods are essential in industrial applications where direct measurements are limited or infeasible. The accuracy and stability of these methods directly influence design decisions, process optimization, and system reliability. Understanding the behaviour and limitations of different numerical approaches is therefore critical in engineering practice.

In this study, the cooling behaviour of a water sample is analysed using experimental measurements, an analytical model, and numerical approximations based on the Euler and fourth-order Runge–Kutta (RK4) methods. The objective is to evaluate the accuracy, stability, and practical applicability of these numerical methods in modelling transient thermal processes. Although the system considered is relatively simple, it serves as a representative case for more complex engineering applications.

MATERIALS AND METHODS

This study combines experimental measurements with analytical modelling and numerical methods to investigate the cooling behaviour of a water sample. The methodology is divided into three main components: experimental data acquisition, analytical modelling, and numerical approximation of the governing differential equation. The symbols used throughout are summarized in Table 1.

Experimental data acquisition

A 50 mL water sample was heated using a 200 W heating mantle from an initial temperature of approximately 27 °C to a maximum temperature of about 55 °C. After reaching the desired temperature, the heating source was removed and the system was allowed to cool under ambient conditions.

Temperature measurements were obtained using a waterproof DS18B20 digital temperature sensor connected to an Arduino Uno microcontroller. The sensor was positioned centrally within the liquid to ensure accurate measurement of the bulk temperature.

Data were recorded at regular intervals of one second and transmitted via serial communication to a computer for storage and further analysis. The recorded temperature–time data were exported and processed for comparison with both analytical and numerical models.

Analytical model

The cooling process was modelled using Newton's law of cooling, expressed as:

$$\frac{dT}{dt} = -k(T - T_\infty) \tag{1}$$

The analytical solution of this equation is given by:

$$T(t) = T_\infty + (T_0 - T_\infty)\, e^{-kt} \tag{2}$$

where T_0 is the initial temperature at the start of the cooling phase (Moran et al., 2014). The value of the cooling constant k was determined by fitting the analytical model to the experimental cooling data.

Numerical methods

To evaluate numerical approximations, the governing differential equation was solved using the forward Euler method and the fourth-order Runge–Kutta (RK4) method. Both approaches approximate the continuous temperature evolution by discretizing time into finite steps of size Δt, where the temperature is updated iteratively based on the governing equation. Two stages were considered: a baseline comparison using a small time step, followed by a systematic investigation of the influence of increasing time step size on numerical accuracy.

The forward Euler method is a first-order scheme in which the temperature is updated using the slope evaluated at the beginning of each time step (Gilat and Subramaniam, 2014):

$$T_{n+1} = T_n + \Delta t\,[-k(T_n - T_\infty)] \tag{3}$$

The fourth-order Runge–Kutta method improves the approximation by incorporating multiple slope evaluations within each time step (Gilat and Subramaniam, 2014):

$$T_{n+1} = T_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{4}$$

where the intermediate slopes are defined as:

$$\begin{aligned} k_1 &= f(T_n) \\ k_2 &= f\left(T_n + \frac{\Delta t}{2} k_1\right) \\ k_3 &= f\left(T_n + \frac{\Delta t}{2} k_2\right) \\ k_4 &= f(T_n + \Delta t\, k_3) \end{aligned} \tag{5}$$

with $f(T) = -k(T - T\infty)$.

Numerical implementation and comparison strategy

All numerical simulations were implemented in Octave, with the governing equation defined as a reusable function and separate functions developed for the Euler and RK4 schemes, allowing modular and reproducible simulations. The cooling process was simulated over a defined time interval using identical initial conditions for both methods, and different time step sizes were tested to evaluate the sensitivity of each method to discretization.

The analysis was structured in two stages. First, a baseline comparison was performed using a small time step (Δt = 1 s), where the experimental data, analytical solution, and numerical methods were compared under conditions of minimal discretization error. Second, the time step size was systematically varied, and for each value of Δt the numerical solutions obtained using the Euler and RK4 methods were compared with the experimental data. The comparison was conducted primarily through graphical analysis of temperature–time curves, with emphasis on the divergence of numerical solutions from the analytical model, the agreement between numerical predictions and experimental data, and the relative sensitivity of each method to increasing Δt.

Computational implementation

The governing equation was implemented as a reusable Octave function (Listing 1), and separate functions were developed for the Euler and RK4 schemes (Listings 2 and 3), enabling modular and reproducible simulations. The main script then called these routines, computed the analytical solution, and compared all results with the experimental data.

Listing 1. Governing equation (coolingODE.m).

```
function dTdt = coolingODE(T, k, T_env)
  % Newton's law of cooling
  dTdt = -k * (T - T_env);
end
```

Listing 2. Forward Euler method (euler_method.m).

```
function [t, T] = euler_method(k, T_env, T0, dt, t_end)
  t = 0:dt:t_end;
  T = zeros(size(t));
  T(1) = T0;
  for i = 1:length(t)-1
    dTdt = coolingODE(T(i), k, T_env);
    T(i+1) = T(i) + dt * dTdt;
  end
end
```

Listing 3. Fourth-order Runge–Kutta method (rk4_method.m).

```
function [t, T] = rk4_method(k, T_env, T0, dt, t_end)
  t = 0:dt:t_end;
  T = zeros(size(t));
  T(1) = T0;
  for i = 1:length(t)-1
    Tn = T(i);
    k1 = -k*(Tn - T_env);
    k2 = -k*((Tn + 0.5*dt*k1) - T_env);
    k3 = -k*((Tn + 0.5*dt*k2) - T_env);
    k4 = -k*((Tn + dt*k3) - T_env);
    T(i+1) = Tn + (dt/6)*(k1 + 2*k2 + 2*k3 + k4);
  end
end
```

Table 1. Symbols and abbreviations used in this study.

Symbol	Description and unit

T	Temperature (°C)
T_0	Initial temperature (°C)
T_∞	Ambient temperature (°C)
t	Time (s)
Δt	Time step size (s)
k	Cooling constant (s^{-1})
T_n	Temperature at step n (°C)
RK4	Fourth-order Runge–Kutta method

RESULTS AND DISCUSSION

The results obtained from the numerical and analytical models are presented and compared with the experimental data. The analysis begins with a baseline comparison for a small time step, followed by an investigation of the influence of time step size on numerical accuracy.

Baseline comparison

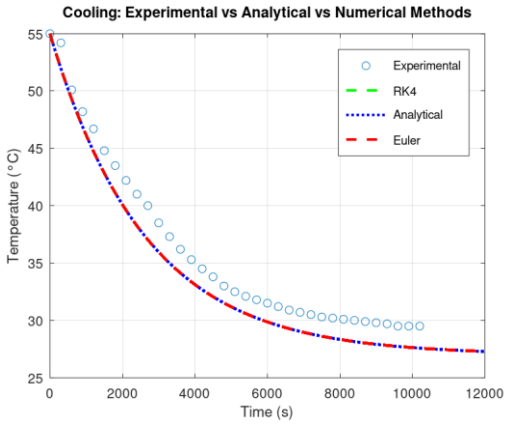

Figure 1. Comparison of experimental data with the analytical solution, Euler method, and RK4 method for a small time step (Δt = 1 s).

Figure 1 shows the comparison between the experimental measurements, the analytical solution of Newton's law of cooling, and the numerical approximations obtained using the Euler and RK4 methods for a small time step (Δt = 1 s). All three model-based approaches follow the general trend of the experimental cooling curve, indicating that Newton's law of cooling provides an appropriate representation of the system. The analytical solution and the RK4 method appear nearly indistinguishable, while the Euler method also closely follows the same trajectory under this fine discretization.

The close agreement between the analytical and numerical solutions is consistent with expectations for small time steps, where discretization error is minimal. Under these conditions, both numerical methods approximate the continuous solution effectively, and differences between methods are not visually significant. Minor deviations between the model predictions and the experimental data are still observed; these may be attributed to heat losses not fully captured by the model, measurement uncertainties, and non-uniform temperature distribution within the liquid. Despite these factors, the overall agreement confirms that the model provides a reasonable approximation of the cooling process.

Influence of time step size on numerical accuracy

To analyse the effect of discretization, the time step Δt was systematically varied and the resulting numerical solutions were compared with the experimental data. For clarity, the results for different time step sizes were plotted within the same graphs, allowing a direct visual comparison of how each method responds to changes in discretization.

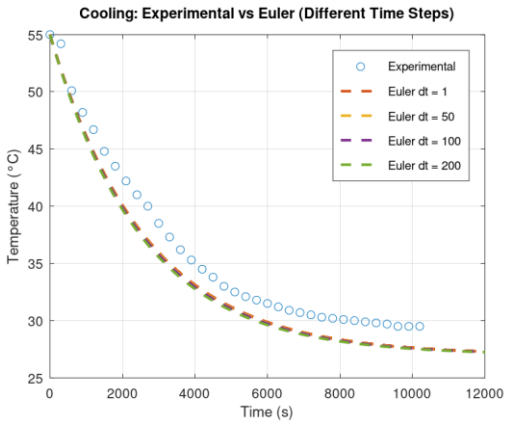

Figure 2. Comparison of experimental data with Euler method predictions for different time step sizes.

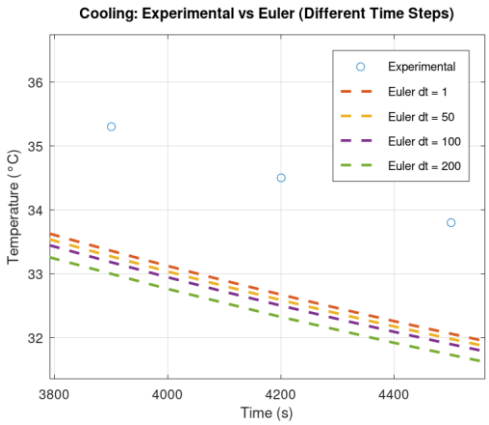

Figure 3. Zoomed-in view highlighting the deviation of Euler method solutions at different time step sizes.

Figure 2 illustrates the effect of increasing the time step size on the Euler method. As Δt increases, the numerical solution deviates progressively from the experimental data, which becomes more apparent in

the zoomed region shown in Figure 3. In the Euler update formula, the temperature change over each step is proportional to both the local slope and the step size Δt. As Δt increases, the increment applied at each step becomes larger, effectively extrapolating the solution over a wider interval using only the slope evaluated at the beginning of the step.

Although the governing differential equation is linear, its solution exhibits exponential decay. Consequently, the temperature slope changes continuously as the system approaches thermal equilibrium. By using only the initial slope over a large interval, the Euler method overestimates the magnitude of the temperature change, resulting in a predicted cooling rate that is faster than the actual physical behaviour. This effect accumulates over successive steps, leading to a systematic deviation from the experimental data: larger values of Δt produce increasingly lower temperature predictions, and the zoomed view confirms that this deviation grows with time, reflecting the cumulative nature of the discretization error.

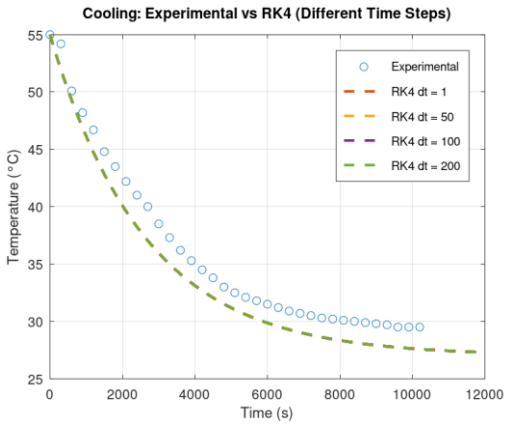

Figure 4. Comparison of experimental data with RK4 method predictions for different time step sizes.

Figure 4 presents the corresponding results for the RK4 method. In contrast to the Euler method, the RK4 solutions remain closely aligned with the experimental data across all tested time step sizes. The RK4 method evaluates the intermediate slopes k_1, k_2, k_3, and k_4 at different points within the step and therefore incorporates information about how the slope evolves over the entire interval, rather than relying solely on the slope at its beginning.

As Δt increases, the RK4 method does not rely on a single extrapolation over a large interval but instead constructs a weighted average of multiple slope evaluations. This allows the method to better approximate the curvature of the underlying solution, even when larger step sizes are used (Gilat and Subramaniam, 2014). Although minor deviations may still occur, the results indicate that the RK4 method is markedly less sensitive to increases in time step size than the Euler method, maintaining a more consistent approximation of the system dynamics under coarser discretization.

Comparison across methods for increasing time step sizes

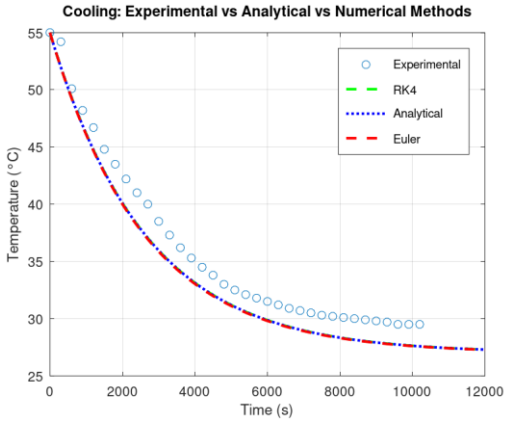

Figure 5. Comparison of all methods for Δt = 50 s.

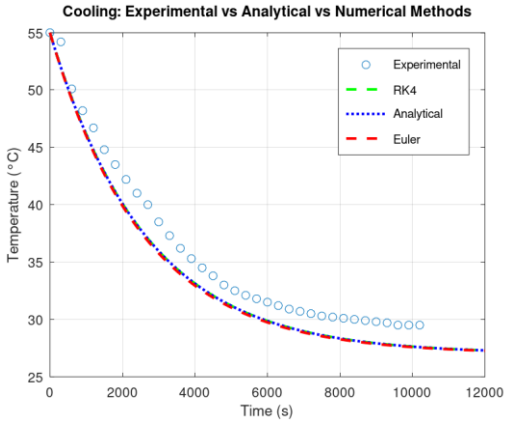

Figure 6. Comparison of all methods for Δt = 100 s.

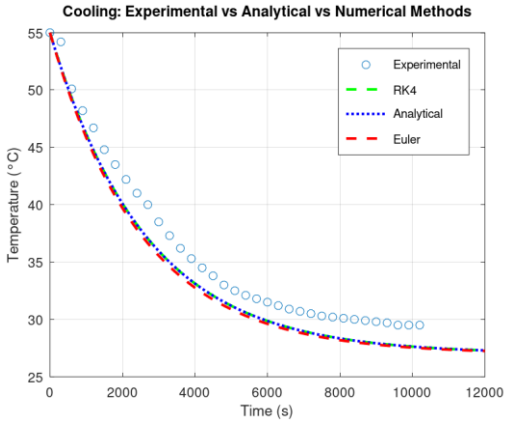

Figure 7. Comparison of all methods for Δt = 200 s.

Figures 5–7 provide a comprehensive comparison of the analytical and numerical methods as the time step increases. The analytical solution represents the continuous, exact solution of the governing differential equation and therefore serves as a

reference for evaluating the numerical approximations, while the experimental data, although subject to physical and measurement limitations, follow a similar overall trend and provide validation of the model.

As observed in the figures, the RK4 method remains closely aligned with the analytical solution across all tested values of Δt, indicating that it preserves the underlying behaviour of the differential equation even under coarser discretization. In contrast, the Euler method progressively deviates from both the analytical and experimental curves as Δt increases. The Euler method, being first-order, has a truncation error proportional to O(Δt), leading to rapid accumulation of error at larger step sizes, whereas the RK4 method is fourth-order, with a truncation error proportional to $O(\Delta t^4)$. By incorporating multiple slope evaluations within each step, RK4 provides a more accurate approximation of the integral of the differential equation over the interval, allowing it to maintain close agreement with the analytical solution even when the discretization becomes relatively coarse.

The comparison with the experimental data supports these observations. While both numerical methods approximate the general cooling trend, the RK4 method remains consistent with both the analytical and experimental results, whereas the Euler method increasingly overestimates the rate of temperature decrease. This indicates that the choice of numerical method has a direct impact on the physical accuracy of the predicted system behaviour.

Overall discussion

The results indicate that the accuracy of the numerical approximation depends strongly on both the chosen method and the time step size. For small values of Δt, both Euler and RK4 produce solutions consistent with the analytical model, suggesting that discretization error is limited under fine time resolution. As Δt increases, the Euler method shows increasing deviation from both the analytical and experimental results, consistent with its first-order formulation, while the RK4 method remains closely aligned with the analytical solution by capturing the evolution of the system more effectively. From a numerical perspective, these results illustrate the influence of method order on stability and accuracy: the RK4 method, although more computationally intensive per step, maintains a more consistent representation of the system behaviour under varying discretization conditions. In practice, higher-order methods may therefore offer advantages when larger time steps are required, at the cost of increased computational effort per iteration.

These observations are summarised in Table 2, which contrasts the principal numerical characteristics of the two schemes evaluated in this study.

Table 2. Qualitative comparison of the Euler and RK4 methods.

Property	**Euler method**	**RK4 method**
Order	First-order	Fourth-order
Truncation error	O(Δt)	$O(\Delta t^4)$
Slope evaluations per step	1	4
Sensitivity to Δt	High	Low
Relative cost per step	Low	Higher

CONCLUSION

In this study, the cooling behaviour of a water sample was analysed using experimental measurements, an analytical solution based on Newton's law of cooling, and numerical approximations obtained through the Euler and fourth-order Runge–Kutta (RK4) methods. Both numerical methods are capable of approximating the analytical solution for sufficiently small time steps. However, the accuracy of the Euler method deteriorates significantly as the time step increases, owing to its first-order formulation and reliance on a single slope evaluation per step, which leads to the accumulation of numerical error and a systematic deviation from both the analytical and experimental results.

In contrast, the RK4 method demonstrates a significantly higher level of accuracy and stability across all tested time step sizes. By incorporating multiple slope evaluations within each step, it provides a higher-order approximation of the governing differential equation, closely matching the analytical solution and maintaining agreement with the experimental data even under coarser discretization. These findings highlight the importance of selecting appropriate numerical methods when modelling transient systems: while simpler methods such as Euler may be suitable for preliminary analysis or very fine discretization, higher-order methods such as RK4 offer clear advantages in accuracy and robustness, particularly where computational efficiency is a key consideration.

Although the system studied here is relatively simple, the insights obtained are directly applicable to more complex engineering problems, where analytical solutions are often unavailable and reliable numerical

approximations are essential. Future work could extend the approach through real-time data integration coupling sensor measurements with the numerical model, the introduction of quantitative error metrics (such as RMSE) and adaptive time-stepping, and the inclusion of additional heat transfer mechanisms such as convection and radiation or spatial temperature variation, yielding a more realistic representation of industrial processes.

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