Stability and numerical analysis of micropolar viscoelastic models

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ABSTRACT

In this work, we consider dynamical systems arising in micropolar viscoelasticity, where the material structure is assumed to have macroscopic and microscopic levels. We show well-posedness by using the theory of linear semigroups. Additionally we analyse the asymptotic behaviour of the solutions where specific polynomial rates of decay are obtained for the solutions of the system. In order to complement the theoretical results, some numerical analysis are applied to the variational version of the system, which includes numerical simulations of the behaviour of the discrete energy.
Palavras-chave: *Viscoelasticity, Micropolar, Stability, Numerical analysis.*

1 Basic Equations

The objective of this section is to propose the mathematical problems that we are going to study in this article. For this purpose it will be convenient to remember that the isotropic and homogeneous micropolar viscoelasticity system is determined by the constitutive equations (see [6]):

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\sigma)e_{ij} + \mu e_{ji} + \lambda_\nu \dot{e}_{rr}\delta_{ij} + (\mu_\nu + \sigma_\nu)\dot{e}_{ij} + \mu_\nu \dot{e}_{ji}, \\ m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + \alpha_\nu\dot{\kappa}_{rr}\delta_{ij} + \beta_\nu\dot{\kappa}_{ji} + \gamma\nu\dot{\kappa}_{ij}, \end{aligned}$$

where

$$e_{ij} = u_{j,i} + \epsilon_{jik}\phi_k, \quad \kappa_{ij} = \phi_{j,i}.$$

It is convenient to remember that t_{ij} is the stress, m_{ij} is the microstress, u_i is the displacement, ϕ_i is the microrotation, $\lambda, \mu, \sigma, \alpha, \beta, \gamma$ are the constitutive constants for the elasticity, while $\lambda_\nu, \mu_\nu, \sigma_\nu, \alpha_\nu, \beta_\nu, \gamma_\nu$ are the viscous coefficients at the macroscopic and microscopic levels, e_{ij} and κ_{ij} are the strain measures, δ_{ij} and ϵ_{ijk} are the usual Kronecker and Ricci symbols.

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The constitutive equations are complemented by the evolution equations

$$t_{ji,j}=\rho\ddot{u}_i \quad \text{and} \quad m_{ji,j}+\epsilon_{ijk}t_{jk}=J\ddot{\phi}_i,$$

where ρ and J are two positive constants whose physical meaning is well known.

The objective of this article is not the study of the system described above, but rather the influence of the dissipation mechanism at the microscopic level on the motion at the macroscopic level (we must assume $\lambda_\nu=\mu_\nu=\sigma_\nu=0$) and the influence of the macroscopic dissipation at the level of micro-rotations. To carry out these tasks, we want to consider more simplified problems and therefore we will restrict ourselves to anti-plane shear deformations in the first case and planar deformations in the second case.

Therefore, in the first case we will study the solutions of the form:

$$u_1=u_2=\phi_3=0,\ u_3(\mathbf{x})=u(x_1,x_2) \quad \text{and} \quad \phi_i(\mathbf{x})=\phi_i(x_1,x_2),\ \ i=1,2.$$

In this case, the previous equations lead us to the system:

$$\rho\ddot{u} = (\mu+\sigma)u_{,jj}-\sigma(\phi_{2,1}-\phi_{1,2}), \tag{1}$$

$$J\ddot{\phi}_1 = \gamma\phi_{1,jj}+(\alpha+\beta)\phi_{j,j1}-\sigma u_{,2}-2\sigma\phi_1+\gamma_\nu\dot{\phi}_{1,jj}+(\alpha_\nu+\beta_\nu)\dot{\phi}_{j,j1}, \tag{2}$$

$$J\ddot{\phi}_2 = \gamma\phi_{2,jj}+(\alpha+\beta)\phi_{j,j2}+\sigma u_{,1}-2\sigma\phi_2+\gamma_\nu\dot{\phi}_{2,jj}+(\alpha_\nu+\beta_\nu)\dot{\phi}_{j,j2}, \tag{3}$$

which can be written as

$$\begin{aligned}\rho\ddot{u}-(\mu+\sigma)\Delta u+\sigma Div(\phi_2,-\phi_1) &= 0,\\ J\ddot{\Phi}-\gamma\Delta\Phi-(\alpha+\beta)\nabla Div\Phi+\sigma\begin{pmatrix}u_y\\-u_x\end{pmatrix}+2\sigma\Phi-\gamma_\nu\Delta\dot{\Phi}-(\alpha_\nu+\beta_\nu)\nabla Div\dot{\Phi} &= 0,\end{aligned} \tag{4}$$

where $\Phi:=(\phi_1,\phi_2)$. We will study this system in a two-dimensional domain B smooth enough to apply the divergence theorem. To propose a well-posed problem in the Hadamard sense, we will need to impose some boundary conditions

$$u(x_1,x_2,t)=0,\ \ \phi_i(x_1,x_2)=0,\ \ i=1,2,\ \ (x_1,x_2)\in\partial B,\ \ t>0, \tag{5}$$

as well as some initial conditions

$$\begin{gathered}u(x_1,x_2,0)=u^0(x_1,x_2),\ \dot{u}(x_1,x_2,0)=v^0(x_1,x_2),\\ \phi_i(x_1,x_2,0)=\phi_i^0(x_1,x_2),\ \ \dot{\phi}_i(x_1,x_2,0)=\psi_i^0(x_1,x_2), i=1,2,\ \ (x_1,x_2)\in B.\end{gathered} \tag{6}$$

It is appropriate to note that if we define the energy of the system as

$$E(t)=\int_B\Big(\rho|\dot{u}|^2+J\dot{\phi}_i\dot{\phi}_i+\mu|\nabla u|^2+\gamma\nabla\phi_i\nabla\phi_i+(\alpha+\beta)\phi_{i,i}\phi_{j,j}+\sigma\phi_i\phi_i+\sigma(|u_x-\phi_2|^2+|u_y+\phi_1|^2)\Big)da,$$

we obtain that

$$E(t)+\int_0^t D(s)ds=E(0),$$

where

$$D(t)=2\int_B\Big(\gamma_\nu\nabla\dot{\phi}_i\nabla\dot{\phi}_i+(\alpha_\nu+\beta_\nu)\dot{\phi}_{i,i}\dot{\phi}_{j,j}\Big)da.$$

In this paper, we will assume that the energy and the dissipation are positive. Therefore, we need to impose that the coefficients

$$\rho,\ J,\ \mu,\ \gamma,\ \alpha+\beta,\ \sigma,\ \gamma_\nu,\ \alpha_\nu+\beta_\nu \tag{7}$$

are positive.

As we have said before, we also study plane deformations with solutions of the form

$$\phi_1 = \phi_2 = u_3 = 0, \;\; \phi_3(\mathbf{x}) = \phi(x_1, x_2), \;\; u_i(\mathbf{x}) = u_i(x_1, x_2), \;\; i = 1, 2.$$

In this case we will assume that the macroscopic structure is dissipative, but the microscopic structure is conservative. That is we assume that $\sigma_\nu = \alpha_\nu = \beta_\nu = \gamma_\nu = 0$. We will study the system

$$\begin{aligned}
\rho \ddot{u}_1 &= (\mu+\sigma) u_{1,jj} + (\lambda+\mu) u_{j,j1} + \sigma \phi_{,2} + \mu_\nu \dot{u}_{1,jj} + (\lambda_\nu + \mu_\nu) \dot{u}_{j,j1}, \\
\rho \ddot{u}_2 &= (\mu+\sigma) u_{2,jj} + (\lambda+\mu) u_{j,j2} - \sigma \phi_{,1} + \mu_\nu \dot{u}_{2,jj} + (\lambda_\nu + \mu_\nu) \dot{u}_{j,j2}, \\
J \ddot{\phi} &= \gamma \phi_{,jj} + \sigma (u_{2,1} - u_{1,2}) - 2\sigma\phi,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\rho \ddot{\mathbf{U}} - (\mu+\sigma)\Delta \mathbf{U} - (\lambda+\mu)\nabla Div \mathbf{U} - \sigma \begin{pmatrix} \phi_y \\ -\phi_x \end{pmatrix} - \mu_\nu \Delta \dot{\mathbf{U}} - (\lambda_\nu + \mu_\nu) \nabla Div \dot{\mathbf{U}} &= 0, \\
J\ddot{\phi} - \gamma \Delta \phi - \sigma Div(u_2, -u_1) + 2\sigma\phi &= 0,
\end{aligned} \tag{8}$$

where $\mathbf{U} = (u_1, u_2)$. Similarly to the problem proposed above, we must impose initial and boundary conditions. Now, it will be

$$\phi(x_1, x_2, t) = 0, \;\; u_i(x_1, x_2) = 0, \;\; i = 1, 2, \;\; (x_1, x_2) \in \partial B, \;\; t > 0 \tag{9}$$

and

$$\begin{gathered}
\phi(x_1, x_2, 0) = \phi^0(x_1, x_2), \;\; \dot{\phi}(x_1, x_2, 0) = \psi^0(x_1, x_2), \\
u_i(x_1, x_2, 0) = u_i^0(x_1, x_2), \;\; \dot{u}_i(x_1, x_2, 0) = v_i^0(x_1, x_2), i = 1, 2, \;\; (x_1, x_2) \in B.
\end{gathered} \tag{10}$$

In this case, the corresponding energy is defined by

$$\begin{aligned}
E(t) &= \frac{1}{2} \int_B \Big(\rho \dot{u}_i \dot{u}_i + J|\dot{\phi}|^2 + \mu \nabla u_i \nabla u_i + (\lambda+\mu) u_{i,i} u_{j,j} + \gamma ||\nabla\phi||^2 + \\
&\qquad + \sigma |u_{1,x}|^2 + \sigma |u_{2,y}|^2 + \sigma(|u_{1,y} + \phi|^2 + |u_{2,x} - \phi|^2) \Big) da,
\end{aligned}$$

we obtain that

$$E(t) + \int_0^t D(s) ds = E(0),$$

where

$$D = \int_B \Big(\mu_\nu \nabla \dot{u}_i \nabla \dot{u}_i + (\lambda_\nu + \mu_\nu) \dot{u}_{i,i} \dot{u}_{j,j} \Big) da.$$

When we consider problem (8)-10, we will impose that the coefficients

$$\rho, \; J, \; \mu, \; \lambda + \mu \, \gamma, \; \sigma, \; \mu_\nu, \; \lambda_\nu + \mu_\nu \tag{11}$$

are positive.

2 Model with microscopic dissipation

In this section we consider the system (4)-(6) and prove wellposedness and polynomial rates of decay for the solutions.

2.1 Existence of Semigroup

The objective of this subsection is to clarify the semigroup structure of contractions that we can see in the case of the solutions proposed for our problem. First we will define the Hilbert space in which we are going to work. We denote by $\mathcal{H}$ the space

$$\mathcal{H} \;=\; H_0^1(B) \times L^2(B) \times \mathbf{H}_0^1(B) \times \mathbf{L}^2(B),$$

where $\mathbf{H}_0^1(B) = H_0^1(B) \times H_0^1(B)$ and $\mathbf{L}^2(B) = L^2(B) \times L^2(B)$ being $L^2(B)$ and $H_0^1(B)$ the usual Sobolev spaces. Here the elements of our space are given by (u, v, Φ, Ψ) with $\Phi = (\phi_1, \phi_2)$ and $\Psi = (\psi_1, \psi_2)$.

If $U^{(i)} = (u^i, v^i, \Phi^i, \Psi^i)$ for $i = 1, 2$ are two elements of the Hilbert space we can define the inner product:

$$\begin{aligned}(U^{(1)}, U^{(2)})_{\mathcal{H}} \;&=\; \rho(v^1, v^2) + \mu((\nabla u^1, \nabla u^2)) + J((\Psi^1, \Psi^2)) + \gamma((\nabla\phi_i^1, \nabla\phi_i^2)) \\ &\quad +(\alpha+\beta)(Div\Phi^1, Div\Phi^2) + \sigma(u_x^1 - \phi_2^1, u_x^2 - \phi_2^2) + \sigma(u_y^1 - \phi_1^1, u_y^2 - \phi_1^2) + \sigma((\Phi^1, \Phi^2)).\end{aligned}$$

This inner product has associated norm

$$||U||_{\mathcal{H}}^2 = \rho|v|^2 + \mu||\nabla u||^2 + J||\Psi||^2 + \gamma((\nabla\phi_i, \nabla\phi_i)) + (\alpha+\beta)|Div\Phi|^2 + \sigma|u_x - \phi_2|^2 + \sigma|u_y - \phi_1|^2 + \sigma||\Phi||^2.$$

It is worth noting that we can write our system by means of the evolution equation

$$\frac{d}{dt}U(t) = \mathcal{A}U, \quad U(0) = (u^0, v^0, \phi_1^0, \phi_2^0, \psi_1^0, \psi_2^0),$$

where the matrix operator is given by

$$\mathcal{A}U = \begin{pmatrix} v \\ -\dfrac{1}{\rho}\Big[-(\mu+\sigma)\Delta u + \sigma Div(\phi_2, -\phi_1)\Big] \\ \Psi \\ -\dfrac{1}{J}\left[-\gamma\Delta\Phi - (\alpha+\beta)\nabla Div\Phi + \sigma\begin{pmatrix} u_y \\ -u_x\end{pmatrix} + 2\sigma\Phi - \gamma_\nu\Delta\Psi - (\alpha_\nu+\beta_\nu)\nabla Div\Psi\right]\end{pmatrix}.$$

The domain of operator $\mathcal{A}$ is the subspace of the elements U such that $\mathcal{A}U \in \mathcal{H}$. We note that it means that

$$\begin{aligned} D(\mathcal{A}) \;&=\; \Big\{u \in \mathcal{H} \;:\; u \in H^2(B) \cap H_0^1(B),\; v \in H_0^1(B),\; \Psi \in \mathbf{H}_0^1(B), \quad \text{with} \\ &\qquad \gamma\Delta\Phi + (\alpha+\beta)\nabla Div\Phi + \gamma_\nu\Delta\Psi + (\alpha_\nu+\beta_\nu)\nabla Div\Psi \in \mathbf{L}^2(B)\Big\}.\end{aligned}$$

Therefore we can claim that it is a dense subspace of our space. Additionally, it is easy to get

$$\mathrm{Re}(\mathcal{A}U, U) = -\gamma_\nu((\nabla\psi_i, \nabla\psi_i)) - (\alpha_\nu+\beta_\nu)|Div\Psi|^2. \tag{12}$$

In view of the previous comments, we can guarantee the existence of a positive constant K such that

$$||U||_{\mathcal{H}} \le K||F||_{\mathcal{H}},$$

which concludes the proof that our operator generates a contractive semigroup and the existence and uniqueness of solutions is also proved.

2.2 Decay to equilibrium of solutions

In the previous section we have seen the existence and uniqueness of solutions as well as the fact that the norm of the solutions does not increase when the time increases. Now, we are going to prove that the solutions tends to zero when the time goes to infinite. To this end we will prove that the imaginary axis is contained at the resolvent of the operator. In fact, let us denote by

$$\mathcal{N} = \{s \in \mathbb{R}^+ :]-is,\, is[\subset \varrho(\mathcal{A})\}.$$

Since $0 \in \varrho(\mathcal{A})$ so we have $\mathcal{N} \neq \emptyset$. Putting $\sigma^* = \sup \mathcal{N}$ we have to possibilities: first $\sigma^* = +\infty$ which implies that $i\mathbb{R} \subseteq \varrho(\mathcal{A})$, or $0 < \sigma^*$ is finite. We will reason by contradiction. Let us suppose that $\sigma^* < \infty$. Then, exists a sequence $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to \sigma^* < \infty$ and

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \quad \longrightarrow \quad \infty.$$

Hence, exists a sequence $\{U_n\} \subset D(\mathcal{A})$ verifying $\|U_n\|_{\mathcal{H}} = 1$ and $\|(i\lambda_n I - \mathcal{A})^{-1} U_n\|_{\mathcal{H}} \to \infty$. That is,

$$i\lambda_n U_n - \mathcal{A}U_n \quad \longrightarrow \quad 0 \quad \text{in} \quad \mathcal{H}, \tag{13}$$

which, in its components, is rewritten as

$$i\lambda u - v \quad \longrightarrow \quad 0 \quad \text{in} \quad H^1(B) \tag{14}$$

$$i\lambda v - (\mu + \sigma)\Delta u + \sigma(\phi_{2,1} - \phi_{1,2}) \quad \longrightarrow \quad 0 \quad \text{in} \quad L^2(B) \tag{15}$$

$$i\lambda \Phi - \Psi \quad \longrightarrow \quad 0 \quad \text{in} \quad \mathbf{H}_0^1(B) \tag{16}$$

$$\begin{aligned} & i\lambda\Psi - \gamma\Delta\Phi - (\alpha + \beta)\nabla(Div\Phi) + \sigma \begin{pmatrix} u_y \\ -u_x \end{pmatrix} + \\ & +2\sigma\Phi - \gamma_\nu \Delta\Psi - (\alpha_\nu + \beta_\nu)\nabla(Div\Psi) \quad \longrightarrow \quad 0 \quad \text{in} \quad \mathbf{L}^2(B), \end{aligned} \tag{17}$$

where we omit the sub-index "n"to simplify the notation. So, using the dissipation equality, we arrive at a contradiction since we have proved that U tends to zero at the norm of the Hilbert space.

2.3 Polynomial Decay of solutions

In this subsection, we show that the solutions of the system (4)-(5) has slow polynomial rates of decay. For this purpose, we will use the following characterization

Let $\{e^{t\mathcal{A}}\}_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. For a fixed $r > 0$ the following conditions are equivalent

(a) There exist $\lambda_0 > 0$ such that

$$||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leq O(|\lambda|^r), \quad \forall |\lambda| \geq \lambda_0.$$

(b) There exist $t_0 > 0$ such that

$$||e^{t\mathcal{A}}\mathcal{A}^{-1}||_{\mathcal{L}(\mathcal{H})} \leq O(t^{-1/r}), \quad \forall t \geq t_0.$$

We point out that the proof of the above result is a direct consequence of Theorem 2.4 in reference [1].

In fact, we note that condition $i\mathbb{R} \subset \varrho(\mathcal{A})$ was already proved in Section 2.2. On the other hand, in order to prove condition (a), note that the resolvent equation

$$(i\lambda I - \mathcal{A}) = F \tag{18}$$

is given by

$$
\begin{aligned}
i\lambda u - v &= f \qquad \text{in} \quad H_0^1(B), & (19)\\
i\lambda\rho v - (\mu+\sigma)\Delta u + \sigma Div(\phi_2,\phi_1) &= \rho g \qquad \text{in} \quad L^2(B), & (20)\\
i\lambda\Phi - \Psi &= \mathbf{H} \qquad \text{in} \quad \mathbf{H_0^1}(B), & (21)\\
i\lambda J\Psi - \gamma\Delta\Phi - (\alpha+\beta)\nabla(Div\Phi) + \sigma\begin{pmatrix} u_y \\ -u_x \end{pmatrix} + & \\
+2\sigma\Phi - \gamma_\nu\Delta\Psi - (\alpha_\nu+\beta_\nu)\nabla(Div\Psi) &= J\mathbf{K} \qquad \text{in} \quad \mathbf{L}^2(B), & (22)
\end{aligned}
$$

where $U=(u,v,\Phi,\Psi)\in D(\mathcal{A})$ with $\Phi=(\phi_1,\phi_2)$, $\Phi=(\psi_1,\psi_2)$ and $F=(f,g,\mathbf{H},\mathbf{K})\in\mathcal{H}$, with $\mathbf{H}=(h_1,h_2)$, $\mathbf{K}=(k_1,k_2)$. Then, multiplying (18) by U in $\mathcal{H}$ and using (12) we obtain

$$
\gamma_\nu((\nabla\psi_i,\nabla\psi_i)) + (\alpha_\nu+\beta_\nu)|Div\Psi|^2 \;\le\; C||U||_{\mathcal{H}}||F||_{\mathcal{H}} \tag{23}
$$

Using multipliers techniques, we deduce

$$
||U||^2 \le C|\lambda|^8||F||^2 \quad\Longrightarrow\quad ||U|| \le C|\lambda|^4||F||, \quad \text{for all } \lambda \text{ large enough.}
$$

Consequently, condition (a) holds for $r=4$. This implies the polynomial stability of the solutions of the system.

2.4 Decay rates improvement: A family of cases

The objective of this subsection is to show a better polynomial decay rates of the solutions of our problem when

$$
\frac{\gamma_\nu}{\gamma} = \frac{\alpha_\nu+\beta_\nu}{\alpha+\beta} =: \xi \neq 0. \tag{24}
$$

To this end, using condition (a), we prove that

$$
\limsup_{|\lambda|\to\infty} \lambda^{-2}||(i\lambda I-\mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty. \tag{25}
$$

The previous arguments imply the polynomial stability with $r=2$ in this case.

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