Estimation of thermal contact conductance in general regular centralized domains by a fractional-order Thikonov's method

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The problem of estimating the thermal contact conductance (TCC) appears in connection with real applications in the heat transfer of composed solids. Examples are given in [?], where two cylindrical tubes are in contact and the goal is to recover internal heat–transfer coefficients from external temperature measurements. A careful and rigorous estimation of the TCC is essential for achieving an accurate understanding of thermal systems, particularly those that reproduce conditions similar to laboratory heat-and-mass transfer experiments. The purpose of this work is to present the mathematical formulation of both the direct and inverse problems for a general regular centralized domain, treating them as instances of a more general transmission problem. Let us consider the steady-state heat transfer between two open sets $\Omega_1 \subset \mathbb{R}^2$ and $\Omega_2 \subset \mathbb{R}^2$ separated by an interface Γ_1, as shown in Figure **??**. Assume that the boundaries Γ_i, $i = 0, 1, 2$ are sufficiently smooth. The outer boundary Γ_2 is subjected to a convective heat transfer with the external ambient at temperature T_∞ and heat transfer coefficient h_2, while the inner boundary Γ_0 is subject to a prescribed temperature T_0. At the contact interface Γ_1, we assume the continuity of the heat flux as well as a Robin boundary condition, where h_1 represents the TCC.

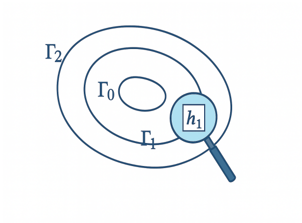

Figure 1: Centralized Domains

Accordingly, the mathematical formulation for this problem reads as

$$\kappa_i \Delta T_i = 0 \qquad \text{in } \Omega_i, \quad i = 1, 2 \tag{1}$$

$$-k_1 \frac{\partial T_1}{\partial n_1} = h_1(T_1 - T_2), \qquad k_1 \frac{\partial T_1}{\partial n_1} = -k_2 \frac{\partial T_2}{\partial n_2} \qquad \text{in } \Gamma_1 \tag{2}$$

$$-k_2 \frac{\partial T_2}{\partial n_2} = h_2(T_2 - T_\infty) \text{ in } \Gamma_2, \qquad T_1 = T_0 \qquad \text{in } \Gamma_0. \tag{3}$$

Here, κ_i, $i = 1, 2$, denote the thermal conductivities of Ω_i; also n_i represents the unit outer normal vector field to Γ_i, $i = 1, 2$. The *forward problem* consists in finding the temperatures T_1,

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and T_2, by solving (??)-(??) from given functions h_1, h_2 and T_0 and ambient temperature T_∞. It is worth noticing that problem (??)-(??) is not a standard transmission problem, as $T_1 \neq T_2$ on the interface Γ_1. Taking this into account, we present a precise definition of weak solution for the forward problem, by assuming that $h_i \in L^2(\Gamma_i)$, $i = 1, 2$, and that $T_0 \in H^{1/2}(\Gamma_0)$. Proceeding further, we establish the uniqueness and existence of weak solutions, by means of the theory of logarithmic potentials as developed in [?]. More precisely, the temperatures T_1 and T_2 are supposed to be a sum of a single and a double logarithmic potential with suitable densities functions:

$$T_1 = S_0(\varphi_0) + K_0(\zeta_0) + S_1(\varphi_1) + K_1(\zeta_1), \qquad T_2 = S_1(\xi) + K_1(\vartheta) + S_2(\varphi_2) + K_2(\zeta_2), \tag{4}$$

where, for $i = 0, 1, 2$,

$$S_i(\varphi_i)(x) = \frac{1}{\pi}\int_{\Gamma_i} \varphi_i(y) \ln\left(\frac{1}{|x-y|}\right) dy, \quad K_i(\zeta_i)(x) = \frac{1}{\pi}\int_{\Gamma_i} \zeta_i(y) \frac{\partial}{\partial n_i} \ln\left(\frac{1}{|x-y|}\right) dy. \tag{5}$$

By considering the limit properties of logarithmic potentials at the boundaries Γ_i, $i = 0, 1, 2$, the problem is then stated as a system of linear equations for the densities φ_i, ζ_i, $i = 0, 1, 2$, and χ, ϑ. The existence and uniqueness of the solutions for such system follows from the Fredholm theory for compact operators.

The *inverse problem* we consider is to recover the TCC (the function h_1) together with the corresponding temperatures T_1 e T_2, from available data T_∞, h_2 and measurements $\mathfrak{T} \in L^2(\Gamma_2)$ of the temperature T_2 on Γ_2. The inverse problem is stated as solving the linear operator equation $B(h_1) = \mathfrak{T}$, where $B : L^2(\Gamma_1) \to L^2(\Gamma_2)$ gives the temperature $T_2|_{\Gamma_2}$ for each given $h_1 \in L^2(\Gamma_1)$. It follows from the representation (??) that the operator B is compact so that the inverse problem is ill-posed. In fact, it is possible to show that the problem is severely ill-posed. As a result, if only noisy data $\widetilde{\mathfrak{T}} \in L^2(\Gamma_2)$ is available, the standard Tikhonov regularization method tends to produce over-smoothing solutions. To overcome such difficulty, we introduce and analyze a regularization method based on the fractional-order Thikonov method [?], which consist in replace the $L^2(\Gamma_2)$-norm by the weighted semi-norm $\|y\|_{\mathfrak{P}} = \|\mathfrak{P}y\|_{L^2(\Gamma_2)}$, where $\mathfrak{P} = (B^*B)^{\frac{(\gamma-1)}{2}}$, for some parameter $0 \leq \gamma \leq 1$. Here $\mathfrak{P}$ relies on the Moore-Penrose pseudoinverse of B^*B whenever $\gamma < 1$. By assuming that $\|\mathfrak{T} - \widetilde{\mathfrak{T}}\| \leq \delta$, $\delta > 0$, approximations h^δ_λ for the sought h_1 are then given as the solution of the following penalized least squares problem:

$$\min_{h \in L^2(\Gamma_1)} \overline{J}_\lambda(h), \qquad \text{where} \qquad \overline{J}_\lambda(h) = \|B(h) - \widetilde{\mathfrak{T}}\|_{\mathfrak{P}} + \lambda\|h\|_{L^2(\Gamma_1)}, \quad \lambda > 0. \tag{6}$$

The regularization parameter $\lambda > 0$ is determined from the discrepancy principle, i.e., we choose $\lambda = \lambda(\delta, \widetilde{\mathfrak{T}})$ such that

$$\|Bh^\delta_\lambda - \widetilde{\mathfrak{T}}\|_{L^2(\Gamma_2)} \leq \tau\delta, \tag{7}$$

where $\tau > 1$ is a user-supplied constant independent of δ.

Keywords: *Ill-posed problems, inverse heat transfer problems, singular integral equations*

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