Amplitude-Based Quantum Encoding for Quantum-Fuzzy Representation in VQC Models using Fuzzy Feature Map

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Abstract— **This paper presents FuzzyFeatureMap, an amplitude-based quantum encoding that maps fuzzy membership degrees to $R_y(\theta)$ rotations for use in Variational Quantum Classifiers (VQC). The approach is compared with Qiskit's `PauliFeatureMap` and `ZZFeatureMap` on a synthesized multi-agent social-decision dataset with five linguistic labels. Using the same ansatz and optimizer, the evaluation reports accuracy, macro F1, training time, and prediction time. FuzzyFeatureMap attains higher accuracy and macro F1 with lower computational cost, indicating benefits from encodings aligned with gradual, uncertain data in quantum machine learning.**

Keywords— **Quantum computing, Fuzzy logic, VQC, Feature map, Quantum encoding, Qiskit.**

I. Introduction

Quantum computing (QC) has become a practical field of research and engineering. Advances in hardware and software now enable flexible algorithms, integrating Fuzzy Logic (FL), which extends classical principles and has a real impact on security, modeling, and artificial intelligence (AI).

Both research areas have been theoretical attempts, using quantum computing to implement fuzzy logic systems faster or more efficiently and developing quantum fuzzy logic systems that integrate quantum mechanics and fuzzy set theory for advanced AI.

Some hybrid or exploratory fields consider quantum fuzzy neural networks, integrating quantum computing with fuzzy logic and neural networks for AI [1] or exploring quantum control systems by using fuzzy logic to manage uncertainty in quantum system controls [2] and applying quantum fuzzy inference systems, which can potentially achieve faster inference using quantum algorithms [3].

Unlike conventional computing with binary bits, quantum computing works with *qubits*, units of information that use *superposition*, *entanglement*, and *interference*. In superposition, a qubit can represent multiple states at once, so the state space grows exponentially as more qubits are added. Entanglement creates correlations that do not appear in classical systems, allowing strong dependencies among variables. Together with control over probability amplitudes, these effects support models that pick up fine-grained patterns and generalize in ways that classical models do not.

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Even with these advances, many real-world datasets are vague or uncertain. *Fuzzy logic* addresses this by using membership degrees, which align with how people describe categories such as "low" and "high."

Fuzzy logic and quantum methods already show up in AI applications, from robotics and healthcare to training and tuning neural networks. This motivates combining them in models that can handle gradual uncertainty. However, feature maps in Qiskit such as `PauliFeatureMap` and `ZZFeatureMap` were built for crisp, classical inputs and do not represent fuzzy semantics well, which can hurt both efficiency and interpretability when uncertainty varies smoothly.

This work considers the *FuzzyFeatureMap*, an encoding that maps fuzzy membership degrees to rotations on qubits. The design preserves continuity and interpretability and integrates naturally into variational quantum models. Experiments show simpler circuits, shorter training time, and equal or better classification performance than standard maps.

II. Preliminaries

A. Basic Concepts of Quantum Computing

QC models information processing based on the principles of quantum mechanics; as a fundamental element, the *qubit* represents a superposition state of the classical basis states $|0\rangle$ and $|1\rangle$: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

A single-qubit state in the three-dimensional representation (Bloch sphere) can be expressed as follows:

$$|\psi\rangle \quad = \quad e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right). \tag{1}$$

The global phase factor $e^{i\gamma}$ is physically irrelevant (unobservable), since it does not change measurement probabilities; note that $|e^{i\gamma}| = 1$. Let $\theta, \phi, \gamma \in \mathbb{R}$. On the Bloch sphere (unit radius), θ is the polar angle from the $+\hat{z}$ axis and ϕ is the azimuthal angle around $\hat{z}$, measured from $+\hat{x}$ toward $+\hat{y}$. This establishes the link between the analytic state representation and its geometric point on the sphere.

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, a measurement in the computational basis yields $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. This process is modeled by projective measurements, which are not unitary gates [4].

The state space of a multi-*qubit* quantum system is obtained by the tensor product of its components. Considering two qubits, $|\psi\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ and $|\varphi\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$, the joint state is described by:

$$|\psi\rangle \otimes |\varphi\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle.$$

The evolution of an n-qubit quantum system is performed by a *unitary operation (gate)*, modeled as a $2^n \times 2^n$ unitary matrix, where n is the number of qubits. In the single-qubit case, a general unitary with $\theta \in [0, \pi]$ and $\phi, \lambda \in [0, 2\pi)$ is described as:

$$F = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}. \tag{2}$$

For parameters $\theta = \frac{\pi}{2}$, $\lambda = \pi$, and $\phi = 0$ in Eq. 2, it yields the *Hadamard gate*:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Applying the operation $H \otimes H$ to the classical state $|01\rangle$ results in a two-qubit quantum state in superposition, relative to the classical basis:

$$|\gamma\rangle = H \otimes H|01\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

So, $|\gamma\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$.

This type of construction is essential for exploring fundamental properties of quantum computing, such as entanglement and interference, applied in quantum learning algorithms and data encoding.

B. Data Encoding and Variational Classification in QML

Quantum machine learning algorithms, such as the *Variational Quantum Classifier* (VQC), combine quantum and classical parts to perform supervised classification. A VQC has three stages:

1) **Encoding (Feature Map):** classical inputs are encoded into quantum states using parameterized gates. This step determines how variables enter the circuit. Two common maps in Qiskit are:
   - **PauliFeatureMap:** applies single-qubit rotations built from Pauli operators; entanglement is optional;
   - **ZZFeatureMap:** adds pairwise ZZ interactions between qubits, which promotes entanglement among features.
2) **Variational circuit (Ansatz):** after encoding, the data pass through a parameterized quantum circuit (the *ansatz*) made of rotations and entangling gates. The circuit parameters are trained to minimize a loss function.
3) **Measurement and Optimization:** at the end of the circuit, one or more qubits are measured and the results are used to compute the predicted output. A classical optimizer, such as SPSA, then updates the ansatz parameters to improve class separation.

This pipeline lets a VQC capture nonlinear relations between variables and use effects like superposition and entanglement to increase the model's expressive power. However, the default feature maps were designed for crisp classical inputs and do not preserve the semantic structure of fuzzy data, meaning that membership degrees and their associated uncertainty.

The study introduces a fuzzy-logic-based encoding used as the feature map within the VQC architecture.

C. Basic Concepts of Fuzzy Logic

Fuzzy logic is a mathematical extension of classical set theory that provides a basis for handling imprecise or uncertain data. Introduced to smooth transitions between classes, fuzzy sets generalize classical set theory, which can be seen as a special case of fuzzy theory [5]. Later many-valued generalizations, formalized in [6], opened the door to applications in several fields.

Classical set theory is based on the characteristic function $f_A \colon U \to \{0,1\}$, where $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$, with U the universe. This function assigns each element $x \in U \neq \emptyset$ a value in the discrete set $\{0,1\}$.

Let $U \neq \emptyset$ be the universe. A fuzzy set A in U is characterized by the membership function $f_A \colon U \to [0,1]$, where, for each $x \in U$, $f_A(x)$ gives the degree to which x belongs to the fuzzy set A.

A fuzzy set A in U can also be described as a set of ordered pairs, with each element $x \in U$ associated with its membership degree $f_A(x) \in [0,1]$, that is,

$$A = \{(x, f_A(x)) \mid x \in U\}.$$

In this context, a fuzzy set can also be defined by n-tuples within a many-valued logic approach.

Let A and B be fuzzy sets in $U \neq \emptyset$, represented by membership functions $f_A, f_B \colon U \to [0,1]$, respectively. With $f_\cup, f_\cap \colon U \to [0,1]$, the union and intersection of A and B are, respectively,

$$A \cup B = \{(x, f_\cup(x)) \mid x \in U\}, f_\cup(x) = \max\{f_A(x), f_B(x)\};$$

$$A \cap B = \{(x, f_\cap(x)) \mid x \in U\}, f_\cap(x) = \min\{f_A(x), f_B(x)\}.$$

The operators $\max, \min \colon [0,1]^2 \to [0,1]$ are triangular norms and conorms and can be replaced by other functions from the same classes, as presented in [7].

Also, let $f_{A'} \colon U \to [0,1]$. The fuzzy set A' expresses the fuzzy complement of A in U under the standard negation $N_S \colon [0,1] \to [0,1]$ given by $N_S(x) = 1 - x$, and given by

$$A' = \{(x, f_{A'}(x)) \mid x \in U\}, \text{ and } f_{A'}(x) = 1 - f_A(x).$$

In this work, *FuzzyFeatureMap* describes an encoding based on $RY(\theta)$ rotations, a standard quantum gate performing a rotation by angle θ around the Y-axis of the Bloch sphere, whenever the angle θ is proportional to the fuzzy membership degree.

In this encoding, the membership degree directly influences the probability of measuring the qubit in the state $|1\rangle$, reflecting fuzzy uncertainty in the quantum amplitude. This approach is simple, efficient, and semantically consistent with the nature of the data.

III. Methodology

Evaluation of the proposed fuzzy quantum encoding considers a Variational Quantum Classifier trained on a dataset synthesized from a computational social dilemma [8].

The related comparison focuses on *FuzzyFeatureMap* versus Qiskit's `PauliFeatureMap` and `ZZFeatureMap`.

The dataset was generated from simulations inspired by social dilemmas with multiple agents. In these simulations, three entities are identified as follows:

- `Police_Officer1`;
- `Police_Officer2`; and
- `Police_Officer3`

They take on different degrees of participation in a collective decision, which can be to predict the prisoner behavior. Each agent contributes a variable action intensity, given by continuous values between 0 and 1, corresponding to fuzzy membership degrees. The system output is a linguistic classification of the collective behavior, categorized as `Very_Low`, `Low`, `Medium`, `High`, or `Very_High`.

Rather than indicating the presence or absence of a behavior in binary form, these values reflect how strongly each agent exhibits a given trait, such as cooperation, alignment, or resistance.

See, e.g., a value of 0.75 for `Police_Officer2` indicates a strong tendency to act or exert influence attributed to that agent in a specific simulation.

The output variable `Label` represents an aggregated fuzzy linguistic classification of the scenario, taking categorical values such as `Very_Low`, `Low`, `Medium`, `High`, and `Very_High`. These labels are derived from aggregated percentages of the simulated decision, reflecting the overall intensity of the collective behavior.

The input values are normalized to $[0, 1]$, and related labels were numerically encoded using `LabelEncoder` for use by the quantum classifier.

A. *Split and Subsampling*

The dataset was split into 80% training and 20% testing with class stratification using the `stratify` parameter of `train_test_split`.

The process to standardize the experiments allows direct comparisons across the tested feature maps, fixing a subsample of 500 training examples was used for all models.

This subsampling reduces the computational cost of training variational quantum circuits while keeping the class representation balanced.

B. *Model*

Experiments used the *Variational Quantum Classifier (VQC)* from the `qiskit-machine-learning` library.

This model architecture considers:

- **Feature map:** three encodings were evaluated — the proposed `FuzzyFeatureMap`, `PauliFeatureMap`, and `ZZFeatureMap`;
- **Ansatz:** `EfficientSU2` with 3 repetitions, combining single-qubit rotations with linear entangling (CX) gates;
- **Optimizer:** `SPSA` with `maxiter = 100`, which is robust to noise and suitable for optimization in quantum parameter spaces.

C. *Evaluation Criteria*

The following metrics are reported for comparison:

- **Accuracy:** proportion of correct predictions over all examples;
- **F1-score (macro):** harmonic mean of precision and recall, averaged uniformly over classes regardless of their frequency;
- **Training time:** time to fit the ansatz parameters on the training set;
- **Prediction time:** time to classify all examples in the test set.

These metrics were chosen to assess not only predictive quality but also computational efficiency, considering practical settings with limited resources.

IV. Results and Discussion

This section presents a comparison of `FuzzyFeatureMap` approach against the standard `PauliFeatureMap` and `ZZFeatureMap` approaches, using a VQC on fuzzy, multi-class data.

The evaluation employed a 500-example training subsample and the full test dataset, ensuring reproducibility across runs.

A. *Quantitative Comparison*

In the experiments, the VQC was trained to classify samples into five linguistic categories: `Very_Low`, `Low`, `Medium`, `High`, and `Very_High`.

Each model input consists of three continuous values in $[0, 1]$ that represent the fuzzy participation degrees of three social agents (e.g., how "friendly" each officer is perceived in a simulated scenario). These non-binary membership degrees express the intensity with which each agent exhibits a given characteristic.

The model output corresponds to the fuzzy classification of the resulting collective behavior; in this case, the subjective probability that a prisoner cooperates with the agents. For example, if all three agents are strongly friendly (high membership values), a high cooperation propensity is expected (classes `High`, `Very_High`). The objective is to predict the aggregated response while preserving gradual semantics and input uncertainty.

During evaluation, a prediction is counted as correct when the predicted label matches the ground-truth category of the sample.

Since the data are imbalanced and the `Medium` class is predominant, the macro F1-score is particularly important for assessing performance across all classes, including minority ones.

Table I shows accuracy, training time, prediction time, and macro F1-score for each evaluated feature map.

TABLE I: Comparison across quantum encoding methods

Feature Map	Acc	MF1	Tr	Pr
FuzzyFeatureMap	**0.76**	**0.37**	**91.61**	**1.48**
PauliFeatureMap	0.74	0.19	162.44	2.42
ZZFeatureMap	0.61	0.28	168.07	2.73

Acc=Accuracy, MF1=Macro F1;
Tr=Training; Pr=Prediction.

The experiments show the proposed `FuzzyFeatureMap` outperforming the traditional methods in several respects. Table I summarizes the results for the three encodings evaluated: `FuzzyFeatureMap`, `PauliFeatureMap`, and `ZZFeatureMap`.

`FuzzyFeatureMap` achieved the highest overall accuracy (**0.76**) and the highest macro F1 (**0.37**), indicating better performance both in terms of correct predictions and balance across classes. It also had substantially lower training time (91.61 s) and prediction time (1.48 s) than the Qiskit feature-map baselines. This suggests that the fuzzy encoding represents the data more directly and in line with its semantics, preserving gradation rather than forcing a crisp-to-qubit mapping.

The model trained with `ZZFeatureMap`, despite posting the lowest overall accuracy (**0.61**), reached a macro F1 of **0.28**, hinting at a slight advantage in accounting for minority classes. This may be related to the use of entanglement in the input encoding, which induces correlations across qubits. However, since `ZZFeatureMap` was designed for crisp classical inputs, its performance degrades in fuzzy settings where category boundaries are fluid and uncertain.

`PauliFeatureMap` attained comparable accuracy (**0.74**) but a much lower macro F1 (**0.19**), indicating a tendency to favor the predominant class (`Medium`) while failing to capture the behavior of less frequent classes. This limitation is expected, since `PauliFeatureMap` applies only independent single-qubit rotations, without promoting entanglement or interactions among features.

These results underscore the importance of using an encoding aligned with the nature of the data: in contexts with uncertainty, gradual variation, or semantic ambiguity, fuzzy–quantum approaches are promising.

B. *Observed Limitations*

Despite the good results, the model with `FuzzyFeatureMap` still struggled with minority classes such as `Very_High` and `Very_Low`, which lowered the macro-averaged F1 score.

This scenario is expected with imbalanced datasets and can be mitigated with resampling techniques or loss-function adjustments in future work.

Observing that similar behavior has been reported before, where VQC showed limits when generalizing to more complex or real data [9].

Difficulty in capturing class diversity was observed for the traditional feature maps, particularly when data were imbalanced and semantically imprecise.

V. Conclusion

This work proposed a fuzzy-logic-based quantum encoding, `FuzzyFeatureMap`. A variational quantum classifier (VQC) was used, and comparisons were made against Qiskit's standard `PauliFeatureMap` and `ZZFeatureMap`.

The results show that `FuzzyFeatureMap` achieved better accuracy, higher macro F1, and shorter runtimes. This supports the idea that encodings which preserve the fuzzy characteristics of the data are better suited to settings with uncertainty, gradual variation, and linguistic representations.

The achieved results reinforce the potential of integrating quantum computing with computational intelligence, especially in uncertainty-sensitive domains. Recent studies identify this convergence as a promising frontier, with challenges and opportunities for innovation [10].

A. *Future Work*

Other interdisciplinary approaches try to combine their strengths, especially in quantum AI and uncertainty modeling based on FL.

Next steps include applying `FuzzyFeatureMap` to additional variational models (supervised and unsupervised) and exploring alternative fuzzy aggregation strategies and custom fuzzy operators for more expressive treatment of imprecise data.

Another direction is to broaden the use of the fuzzy encoding in hybrid architectures that integrate classical and quantum layers, allowing more flexible modeling.

Finally, the impact of fuzzy encoding in deeper parameterized quantum circuits (deep quantum circuits) will be examined, particularly alongside noise-mitigation techniques.

These directions outline fertile ground for integrating quantum computing and computational intelligence where uncertainty is intrinsic to the data.

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