Zero-error Capacity and Non-Ergodic Quantum Channels

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***Abstract*— In this paper, we investigate preliminary connections between zero-error capacity and non-ergodic quantum channels. We begin by showing that every non-ergodic quantum channel possesses a positive zero-error capacity. Under additional assumptions, we further demonstrate that the adjoint channel also exhibits a positive zero-error capacity. Finally, we conclude by discussing the relationship between zero-error capacity and the projectors fixed by the non-ergodic unital channel and its adjoint.**

***Keywords*— Quantum channel, Zero-error capacity, Non-ergodic.**

I. Introdução

In quantum information theory, the most general representation of a (possibly open) quantum system is given by a quantum channel, that is, a completely positive, trace-preserving linear map acting on the system's states. The qualitative behavior of such channels provides valuable insights into the underlying dynamics of the quantum system. An important example is the class of ergodic quantum channels, namely those that admit a unique fixed point in the space of density matrices. The study of ergodicity in quantum channels has attracted significant attention and has found applications across several areas of quantum information theory [1], [2], [3].

In addition, another relevant class of quantum channels is formed by those admitting at least two distinct fixed points, which we refer to as non-ergodic quantum channels. This class plays an important role in the study of dynamical semigroups [1]. In this article, we employ the notion of non-ergodic quantum channels to explore preliminary connections between this property and the zero-error capacity of such channels.

In quantum information theory [4], the study of the zero-error capacity of quantum channels has emerged as an important line of research [5], [6], [7]. The zero-error capacity was introduced by Medeiros and Assis [5] as the maximum rate at which classical information can be transmitted through a quantum channel with an error probability exactly equal to zero. In this framework, quantum block coding is employed: classical information is encoded into quantum states, transmitted through a discrete memoryless channel (DMC), and subsequently measured at reception. Under these conditions, the transmission occurs with an error rate precisely equal to zero. The zero-error capacity of quantum channels thus generalizes the notion of zero-error capacity for DMCs, originally introduced by Shannon [8].

The zero-error capacity of a channel is related to the number of its fixed points (Proposition 2), in the sense that it is bounded above by the logarithm of the number of such fixed points. This observation motivates the investigation of the relationship between zero-error capacity and non-ergodic quantum channels.

In this context, in order to pursue these investigations, we revisit the mathematical concept of invariant subspaces of a linear operator (or matrix), as this provides a useful framework for understanding the notion of invariant subspaces of a quantum channel. Moreover, for the purposes of our analysis, it is important to emphasize the close relationship between fixed points and the invariant subspaces of a channel [9], [1].

To achieve the main objective of this article, the paper is organized as follows. In Section II, we introduce elementary definitions and properties of quantum channels. Section III is devoted to the definition of zero-error capacity and a discussion of some of its key properties. In Section IV, we present the concept of invariant subspaces of quantum channels and establish results that are central to our analysis. Section V contains the main contributions of this work, where we examine the zero-error capacity of non-ergodic quantum channels. Finally, in Section VI, we summarize our findings and outline preliminary conclusions.

II. Definitions and Quantum Channel

Let $\mathcal{H}$ be a Hilbert space of dimension d. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$, which forms a Hilbert space of dimension d^2.

A quantum channel on the Hilbert space $\mathcal{H}$ is a linear map that is completely positive and trace-preserving (CPTP), acting on the set of density operators. Such a channel is denoted by $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$. By the Kraus representation theorem, there exists a collection of Kraus operators $\{A_i\}_{i=1}^{\kappa} \in \mathcal{B}(\mathcal{H})$ such that the action of $\mathcal{E}$ can be expressed as

$$\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^\dagger \text{ and } \sum_{i=1}^{\kappa} A_i^\dagger A_i = I, \tag{1}$$

for all $\rho \in \mathcal{B}(\mathcal{H})$.

Since $\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^\dagger$ is a quantum channel with Kraus representation $\{A_i\}_{i=1}^{\kappa}$, we define the adjoint of $\mathcal{E}$, denoted by $\mathcal{E}^\dagger$, as the quantum channel represented by the

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adjoints of the Kraus operators, that is,

$$\mathcal{E}^{\dagger}(\rho) = \sum_{i=1}^{\kappa} A_i^{\dagger} \rho A_i, \tag{2}$$

for all $\rho \in \mathcal{B}(\mathcal{H})$.

It is also important to note that $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ have identical spectra, i.e., they share the same eigenvalues.

III. Quantum Zero-Error Capacity

Let $\mathcal{X} \subset \mathcal{H}$ be the set of possible input states for the quantum channel $\mathcal{E}$. If $\rho \in \mathcal{X}$, we denote by $\sigma = \mathcal{E}(\rho)$ the quantum state received when ρ is transmitted through the quantum channel, which can be written as

$$\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^{\dagger}. \tag{3}$$

Let $\mathcal{E}$ be a quantum channel. The communication protocol associated with the zero-error capacity can be summarized as follows. Define $\mathcal{S} = \{\rho_1, \ldots, \rho_\ell\} \subset \mathcal{X}$ as a finite subset, and let $\rho_i \in \mathcal{S}$. The states ρ_i form the alphabet of a zero-error quantum code.

The set of codewords of length n is a superset of the sequences formed by n-fold tensor products of states from $\mathcal{S}$, denoted by $\mathcal{S}^{\otimes n}$. For a sequence $\rho_{i_1}, \ldots, \rho_{i_n}$, the i-th codeword is given by $\overline{\rho}_i = \rho_{i_1} \otimes \cdots \otimes \rho_{i_n}$. If Bob performs measurements using a POVM (Positive Operator-Valued Measurement) $\{M_j\}$, satisfying $\sum_j M_j = I$, then we define $p(j|i)$ as the probability that Bob observes outcome j given that Alice sent the state ρ_i. Thus,

$$p(j|i) = \text{tr}[\sigma_i M_j] = \text{tr}[\mathcal{E}(\rho_i) M_j]. \tag{4}$$

A quantum (m, n) zero-error code for $\mathcal{E}$ is composed of:

1) A set of indices $\{1, ..., m\}$, where each index is associated with a classical message;
2) A coding function
$$f_n : \{1, ..., m\} \longrightarrow \mathcal{S}^{\otimes n} \tag{5}$$
which takes each codeword and returns $f_n(1) = \overline{\rho_1}, ..., f_n(m) = \overline{\rho_m}$.
3) A decoding function
$$g : \{1, ..., k\} \longrightarrow \{1, ..., m\} \tag{6}$$
which deterministically associates a message with one of the possible measurement $y \in \{1, ..., k\}$ performed by the POVM $\{M_i\}_{i=1}^{k}$. Furthermore, the decoding function has the following property:
$$\text{Pr}[g(\mathcal{E}(f_n(i))) \neq i] = 0 \tag{7}$$
for all $i \in \{1, ..., m\}$.

The rate of a (m, n) code is given by

$$R = \frac{1}{n} \log\ m \ \ \text{bits/use}. \tag{8}$$

Definition 1 (Quantum Zero-Error Capacity [5]): The quantum zero-error capacity of a quantum channel $\mathcal{E}(\cdot)$, denoted by $C^{(0)}(\mathcal{E})$, is the supremum of the achievable rates with decoding error probability equal to zero,

$$C^{(0)}(\mathcal{E}) = \text{sup}_{\mathcal{S}} \text{sup}_n\ \frac{1}{n} \text{log}\ m, \tag{9}$$

where m is the maximum number of classical messages the system can transmit without error when a zero-error quantum block code (m, n) is used and the input alphabet is $\mathcal{S}$.

A fundamental property of quantum states is distinguishability. Two quantum states are distinguishable if, and only if, the Hilbert subspaces generated by the supports of these states are orthogonal. Thus, given two quantum states $|\rho_i\rangle, |\rho_j\rangle \in \mathcal{S}$ with $i \neq j$, we say that $|\rho_i\rangle$ and $|\rho_j\rangle$ are *non-adjacent* (or distinguishable) at the output of the quantum channel $\mathcal{E}$ if $\mathcal{E}(|\rho_i\rangle)$ and $\mathcal{E}(|\rho_j\rangle)$ belong to orthogonal Hilbert subspaces. Otherwise, $|\rho_i\rangle$ and $|\rho_j\rangle$ are said to be *adjacent* (or indistinguishable) at the output of $\mathcal{E}$. Accordingly, the zero-error capacity conditions in [5], [10] (see Proposition 1 and Proposition 3, respectively) establish that a quantum channel $\mathcal{E}$ has positive zero-error capacity if, and only if, there exist at least two non-adjacent quantum states.

There is a relationship between the zero-error capacity and the fixed points of the quantum channel $\mathcal{E}$. Schauder's fixed point theorem guarantees that every quantum channel has at least one quantum state ρ that is a fixed point of $\mathcal{E}$, i.e., $\mathcal{E}(\rho) = \rho$. This relationship is formalized in the following proposition.

Proposition 2: Let $\mathcal{E}$ be a quantum channel with N_f fixed points. Then the zero-error capacity of $\mathcal{E}$ is at least $log\ N_f$.

The proof of this result can be found in [5], Proposition 2.

IV. Invariant subspace of a quantum channel

Let $W \subseteq \mathcal{H}$ be a vector subspace. We say that W is an invariant subspace of an operator $A \in \mathcal{B}(\mathcal{H})$, or A-*invariant*, if $A|\nu\rangle \in W$ for all $|\nu\rangle \in W$ [9]. Furthermore, we say that W is a common invariant subspace for a set of operators $A_1, \ldots, A_s \in \mathcal{B}(\mathcal{H})$ if W is A_i-invariant for all $i = 1, \ldots, s$, i.e., $A_i|\nu\rangle \in W$ for all $|\nu\rangle \in W$.

In studies of invariant subspaces of linear operators, the zero subspace and the entire Hilbert space $\mathcal{H}$ are always invariant and are referred to as trivial invariant subspaces. Any other invariant subspaces are called non-trivial invariant subspaces. In the context of quantum information theory, the concept of invariant subspaces for linear operators can be naturally extended to quantum channels.

Definition 3 (Inv. Subspace of a Quantum Channel [1]): Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel represented by Kraus operators $A_1, \ldots, A_\kappa$. A subspace $W \subseteq \mathcal{H}$ is said to be invariant under the quantum channel $\mathcal{E}$ if it is a common invariant subspace for all operators A_i, with $i = 1, \ldots, \kappa$, i.e., $A_i|\nu\rangle \in W \quad$ for all $|\nu\rangle \in W$.

In the definition of an invariant subspace of a quantum channel represented via Kraus operators A_i, it is important to note that the invariance of a subspace $W \subseteq \mathcal{H}$ under the quantum channel $\mathcal{E}$ depends on W being a common invariant subspace for all operators A_i, with $i = 1, \ldots, \kappa$. The condition that W is an invariant subspace for the quantum channel $\mathcal{E}$ can be equivalently expressed by the following properties [11], [1]:

1) $A_i P = P A_i P$ for $i = 1, ..., s$ and P is a projector onto W;
2) $\mathcal{E}(P) = P\mathcal{E}(P)P$;
3) $\text{Supp}[\mathcal{E}(\rho)] \subseteq W$ for all states ρ with $\text{Supp}(\rho) \subseteq W$ ($\text{Supp}(\rho)$ represents the support of ρ).

Lemma 4 ([1]): Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$, thus

$$\rho = q\sigma + (1-q)\tau, \tag{10}$$

where $q \in (0, 1]$ and $\tau \in \mathcal{B}(\mathcal{H})$ if, and only if, $\text{Supp}(\sigma) \subseteq \text{Supp}(\rho)$.

A proof of this result can be found in [1, Lemma 8].

Proposition 5 ([1]): If ρ is a fixed point of $\mathcal{E}$, then the subspace generated by $\text{Supp}(\rho)$ is an invariant subspace of $\mathcal{E}$. Furthermore, if $W \subset \mathcal{H}$ is an invariant subspace of $\mathcal{E}$, then there exists a fixed point $\rho_W \in W$ such that $\text{Supp}(\rho_W) \subset W$.

Proof: Given $|\psi\rangle \in \text{Supp}(\rho)$, then by Lemma 4 there exists a probability $p > 0$ and a state σ such that

$$\rho = p\,|\psi\rangle\langle\psi| + (1-p)\sigma. \tag{11}$$

Thus, by applying the quantum channel $\mathcal{E}$ to Eq. (11) and using linearity, we obtain

$$\mathcal{E}(\rho) = p\mathcal{E}(|\psi\rangle\langle\psi|) + (1-p)\mathcal{E}(\sigma). \tag{12}$$

By Lemma 4, we have

$$\text{Supp}[\mathcal{E}(|\psi\rangle\langle\psi|)] \subseteq \text{Supp}[\mathcal{E}(\rho)]. \tag{13}$$

Now, let ρ be a fixed point of $\mathcal{E}$, so that $\mathcal{E}(\rho) = \rho$. From Eq. (13), it follows that

$$\text{Supp}[\mathcal{E}(|\psi\rangle\langle\psi|)] \subseteq \text{Supp}(\rho). \tag{14}$$

The conclusion is that $\text{Supp}(\rho)$ is an invariant subspace of $\mathcal{E}$. Conversely, if W is an invariant subspace of $\mathcal{E}$, then by restricting the input states to those in W, denoted by $\mathcal{E}|_W$, we see that $\mathcal{E}|_W$ is itself a quantum channel in $\mathcal{C}(W)$. Therefore, $\mathcal{E}|_W$ has a fixed point $\rho_W \in W$ such that $\mathcal{E}|_W(\rho_W) = \rho_W$.

Using the first part of this proposition together with the equivalent property (3) of Definition 3, we can conclude that

$$\text{Supp}(\rho_W) \subset W.$$

■

V. Zero-error capacity and non-ergodic quantum channels

In this section, we present some preliminary results concerning the zero-error capacity of non-ergodic quantum channels. The first result establishes that every non-ergodic quantum channel possesses a positive zero-error capacity.

Definition 6: Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel. We say that $\mathcal{E}$ is non-ergodic if it has at least two distinct fixed points.

Theorem 7: Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a non-ergodic quantum channel. Then the zero-error capacity of $\mathcal{E}$ is positive.

Proof: By hypothesis, $\mathcal{E}$ is a non-ergodic quantum channel. Thus, by definition, $\mathcal{E}$ has at least two distinct fixed points. According to Proposition 2, the zero-error capacity of a quantum channel is lower bounded by the logarithm of the number of its distinct fixed points. Therefore, for a non-ergodic channel $\mathcal{E}$, the zero-error capacity satisfies

$$C^{(0)}(\mathcal{E}) \geq \log 2.$$

■

The converse of Theorem 7 does not hold; that is, there exist quantum channels with positive zero-error capacity that have a single fixed point. An example illustrating this is provided in the following.

Example 8: [6] Let a quantum channel $\mathcal{E}$ be represented by the following Kraus operators:

$$A_1 = \alpha(|00\rangle\langle 00| + |11\rangle\langle 11|) + |01\rangle\langle 01| + |10\rangle\langle 10| \tag{15}$$

$$A_2 = \beta(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 01| + |10\rangle\langle 10|) \tag{16}$$

$$A_3 = \beta(|00\rangle\langle 00| + |11\rangle\langle 11| - |01\rangle\langle 01| - |10\rangle\langle 10|), \tag{17}$$

where γ is a scalar such that $0 < \gamma < 1$, and $\alpha = \sqrt{1-2\gamma}$, $\beta = \sqrt{\gamma/2}$.

It can be observed that $\mathcal{E}(I) = \sum_{i=1}^{3} A_i A_i^\dagger \neq I$, and therefore, this channel is not unital. Furthermore, in this channel model, there exists only one state ρ such that $\mathcal{E}(\rho) = \rho$ (see Example 6.2 in [6]).

The zero-error capacity of this channel is positive. Indeed, by choosing the states $|00\rangle = (1, 0, 0, 0)^T$ and $|10\rangle = (0, 0, 1, 0)^T$, we have

$$\text{tr}\big(\mathcal{E}(|00\rangle\langle 00|)\mathcal{E}(|10\rangle\langle 10|)\big) = 0,$$

and thus the zero-error capacity of this quantum channel is positive.

The following result provides a condition under which a quantum channel can simultaneously possess a positive zero-error capacity and be non-ergodic.

Theorem 9: Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a quantum channel such that $W_1 \neq \{0\}$ and $W_2 \neq \{0\}$ are invariant subspaces of $\mathcal{E}$ with $W_1 \cap W_2 = \{0\}$. Then, $\mathcal{E}$ has positive zero-error capacity and is also non-ergodic.

Proof: Since $W_1 \neq \{0\}$ and $W_2 \neq \{0\}$ are invariant subspaces of $\mathcal{E}$, it follows from the second part of Proposition 5 that there exist states $\rho_{W_1} \in W_1$ and $\rho_{W_2} \in W_2$ which are fixed points of $\mathcal{E}$.

Thus, the zero-error capacity of $\mathcal{E}$ satisfies $C^{(0)}(\mathcal{E}) > \log 2$. Furthermore, since $\mathcal{E}$ has at least two fixed points, it follows that $\mathcal{E}$ is non-ergodic. ■

Let $\mathcal{E}$ be a quantum channel represented by Kraus operators $A_1, \ldots, A_\kappa$. We denote the noise commutant [11] of $\mathcal{E}$ by $\mathcal{A}'$ and define it as the set of all operators in $\mathcal{B}(\mathcal{H})$ that commute with both A_i and $A_i^\dagger$, i.e.,

$$\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}) : [\rho, A] = 0 = [\rho, A^\dagger], \forall A \in \{A_i\}_{i=1}^{\kappa}\}. \tag{18}$$

For unital channels, i.e., quantum channels that fix the identity operator ($\mathcal{E}(I) = I$), every $\rho \in \mathcal{A}'$ satisfies $\mathcal{E}(\rho) = \rho$. Similarly, the adjoint channel $\mathcal{E}^\dagger$ also fixes the identity ($\mathcal{E}^\dagger(I) = I$) [1], and thus for all $\rho \in \mathcal{A}'$, we have $\mathcal{E}^\dagger(\rho) = \rho$. Since $\rho \in \mathcal{A}'$, then

$$\mathcal{E}(\rho) = \sum_{i=1}^{\kappa} A_i \rho A_i^\dagger = \sum_{i=1}^{\kappa} \rho A_i A_i^\dagger = \sum_{i=1}^{\kappa} \rho \mathcal{E}(I) = \rho \tag{19}$$

and

$$\mathcal{E}^\dagger(\rho) = \sum_{i=1}^{\kappa} A_i^\dagger \rho A_i = \sum_{i=1}^{\kappa} \rho A_i^\dagger A_i = \sum_{i=1}^{\kappa} \rho \mathcal{E}^\dagger(I) = \rho. \quad (20)$$

On the other hand, the fixed points of the unital channels $\mathcal{E}$ and $\mathcal{E}^\dagger$ belong to $\mathcal{A}'$ [12]. In other words, the fixed points of $\mathcal{E}$ and $\mathcal{E}^\dagger$ coincide.

Theorem 10: Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a unital and non-ergodic quantum channel. Then, the adjoint channel $\mathcal{E}^\dagger$ also has a positive zero-error capacity.

Proof: By hypothesis, the channel $\mathcal{E}$ is unital. Therefore, from Equations (19) and (20), we have that $\mathcal{E}$ and $\mathcal{E}^\dagger$ share the same fixed points. Since $\mathcal{E}$ is non-ergodic, it has at least two fixed points, and the same holds for its adjoint channel $\mathcal{E}^\dagger$.

Hence, by Proposition 2, we conclude that the zero-error capacity of the adjoint channel $\mathcal{E}^\dagger$ is positive. ■

VI. Conclusions

In this paper, we explored some preliminary relationships between zero-error capacity and non-ergodic quantum channels. We showed that every non-ergodic quantum channel has a positive zero-error capacity. Additionally, under the hypothesis that the channel is unital, we demonstrated that the adjoint channel also possesses a positive zero-error capacity.

Furthermore, we highlighted the direct relationship between the zero-error capacity and the number of projectors fixed by the non-ergodic unital channel and by its adjoint. Finally, as significant as the results presented here, this work opens the possibility for exploring further connections between the concepts of zero-error capacity and the class of non-ergodic quantum channels.

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