Precision bounds for coherent transport

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***Abstract*— Heat dissipation is a major hurdle in the design of viable and scalable quantum computers. While the discussion of heat dissipation in computation is normally dominated by the consideration of Landauer's Principle, where irreversible computations inevitably dissipate heat, controlling the fluctuations of the currents needed to operate such a computer would also inevitably entail heat dissipation, as demonstrated by the Thermodynamic uncertainty relations (TURs). These represent an important recent development in nonequilibrium thermodynamics, allowing one to place fundamental lower bounds on the noise-to-signal ratio (precision) of currents in nanoscale devices and showing that these precision limits are intimately connected with the entropy production in the device. Originally formulated for classical time-homogeneous Markov processes, these relations were shown to be violated in the case of quantum-coherent thermoelectric transport. In this work we derive alternative bounds for precision that while still following the spirit of the original TUR bound, remain valid for thermoelectrics. A comparison with other known bounds is presented that makes it clear that this new bound works in a complementary fashion with the known bounds.**

***Keywords*— Quantum thermodynamics, Thermoelectric transport, Signal to noise ratios**

I. INTRODUCTION

When one thinks of a macroscopic thermodynamical system, the main quantities of interest can all be expressed as well-behaved averages, even when we move beyond equilibrium states and consider non-equilibrium steady states. However, when considering a micro or a nano scale device, then the fluctuations become just as important as the averages. In particular, for the design of such devices we want to keep the fluctuations under control so that the system remains operating in the regime it was designed for, with failure to do so potentially leading to damage to the device.

The Thermodynamic uncertainty relation (TUR) is a fairly recent result concerning the fluctuations of currents in a non-equilibrium steady state [1]:

$$\mathrm{Var}(j) \geq \frac{2j^2}{\Sigma} \tag{1}$$

where j is the current intensity, $\mathrm{Var}(j)$ are the current fluctuations and Σ is the rate of entropy production. The interesting point of equation (1) is that it entails a trade-off between the signal to noise ratio of the current and the entropy production (and hence the heat dissipated by the transport):

$$\Sigma \geq \frac{2j^2}{\mathrm{Var}(j)} = 2\mathcal{S}_j \tag{2}$$

where $\mathcal{S}_j$ is the signal to noise ratio of the current. In this sense, TURs provide a limitation to how much we can reduce the heat dissipation in a system, similar to the case of Landauer erasure and therefore may impose some restrictions on the design of quantum computers.

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A. TURs for thermoelectrics in the Landauer-Buttiker regime

The bound in equations (1) and (2) was originally derived for classical systems and it was proven early on that, for quantum systems, bosonic transport obeys this TUR but violations were possible in fermionic transport [2], [3]. Recently, some bounds were found for the fluctuations of the particle current in thermoelectric transport in the Landauer-Buttiker regime.

This regime corresponds to the case where the electrons don't interact with each other and are scattered elastically by some system between two terminals. These will be called left and right terminal and have respectively temperatures $1/\beta_L$, $1/\beta_R$ and chemical potentials μ_L, μ_R. In this case, the transport can be entirely described in terms of a transmission function $\mathcal{T}(\varepsilon)$ (ranging from 0 to 1). A general current and its fluctuations have the form

$$j_h = \int \mathcal{T}(\varepsilon)h(\varepsilon)\Delta f(\varepsilon)d\varepsilon \tag{3}$$

$$\mathrm{Var}(j_h) = \int h(\varepsilon)^2 \left(\mathcal{T}(\varepsilon)g(\varepsilon) + \mathcal{T}(\varepsilon)(1-\mathcal{T}(\varepsilon))\Delta f(\varepsilon)^2\right) d\varepsilon \tag{4}$$

where $\Delta f = f_L - f_R$ is the difference of the Fermi occupations ($f_\alpha = 1/\big(1+e^{\beta_\alpha(\varepsilon-\mu_\alpha)}\big)$) between the left and right terminals and $g = f_L(1-f_L) + f_R(1-f_R)$ is a function of these occupations. Some important examples are the particle current I that corresponds to the case $h(\varepsilon) = 1$ and the rate of entropy production that corresponds to $h(\varepsilon) = \delta_{\beta\mu} - \delta_\beta\varepsilon$, where $\delta_\beta = \beta_L - \beta_R$ and $\delta_{\beta\mu} = \beta_L\mu_L - \beta_R\mu_R$ are the gradients driving the transport.

The bounds we will be comparing ours with are

- $\dfrac{\mathrm{Var}(I)}{|I|} \geq \mathrm{csch}\left(\dfrac{\Sigma}{2|I|}\right)$ (found by Brandner et al [4])
- $\mathrm{Var}(I) \geq \Delta^2_{\mathrm{boxcar}}(I,\Sigma)$, where $\Delta^2_{\mathrm{boxcar}}(I,\Sigma)$ is the variance obtained by the unique boxcar transmission function that induces a particle current I and entropy production rate Σ, while having a support of the form $g(\varepsilon) \leq (\lambda\varepsilon+\mu)\Delta f(\varepsilon)$ (found by Timpanaro et al [5])

II. OUR TUR BOUND

The bound in our work is obtained by changing the constant in the classical TUR from an universal value to a value c that depends on the terminals (but not on other details of the transport):

$$\mathrm{Var}(j) \geq \frac{cj^2}{\Sigma} \Leftrightarrow \Sigma \geq c\,\mathcal{S}_j \tag{5}$$

The constant c is obtained by the following minimization:

$$c = \min_{\varepsilon}\left\{\frac{(\delta_{\beta\mu}-\delta_\beta\varepsilon)g(\varepsilon)}{\Delta f(\varepsilon)}\right\} \tag{6}$$

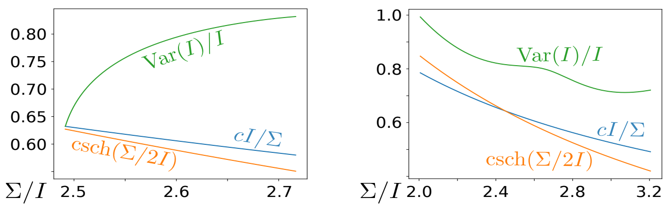
Conceptually this means that we can restore a bound in the same vein as equation (1), but with a constant that depends on the temperatures and potentials involved between the 2 terminals. From a mathematical point of view equation (5) follows by combining (3), (4) and the definition (6) in a Cauchy-Schwarz inequality.

III. Comparison with known bounds

In order to draw a comparison between the bounds we just presented, we will need to focus on bounding the variance of the particle current I in a situation where the average I and the entropy production are known. To interpret this comparison, it is also useful to provide the context for each of these bounds:

- The boxcar bound $B_{\text{box}} \equiv \Delta^2_{\text{boxcar}}(I, \Sigma)$ is derived by finding the transmission function with the smallest variance for fixed I and Σ. As such it is in a sense the tightest bound possible. However the numerical procedure to find the actual value of the bound is fairly complicated and any insights it provides on the relation between fluctuations, current intensity and entropy production can only be obtained indirectly.
- The Brandner bound $B_{\text{Br}} \equiv |I| \operatorname{csch}(\Sigma/2|I|)$ on the other hand is looser but it has a much simpler expression. It also has the advantage of being independent on the gradients driving the transport.
- Our bound $B \equiv cI^2/\Sigma$ is also given by a simple expression, that attempts to recover the clear interpretation of the tradeoff between fluctuations, current intensity and entropy production found in the original TUR. One downside is that the constant c does depend on the gradients (which is actually necessary, since it can be easily shown that increasing the gradients can lead to violations for any fixed $c > 0$ [6])

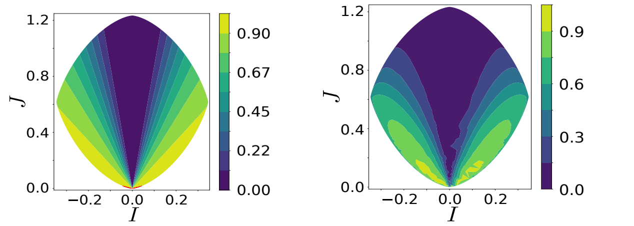
Comparing our bound with the Brandner bound, there are regimes where our bound is tighter and regimes where it is looser. Figure 1 shows an example of this for $\beta_L = 2, \beta_R = 3, \mu_L = 1, \mu_R = 0$.



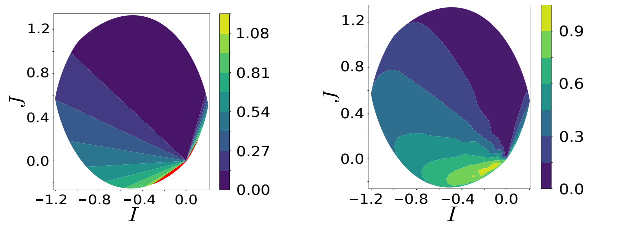
(a) $\mathcal{T} = \mathcal{B}(\varepsilon_m - a, \varepsilon_m + a)$ (b) $\mathcal{T} = \mathcal{B}(\varepsilon_m - a, \varepsilon_m + W - a)$

Fig. 1: Comparison between our bound (B in blue), the bound found in [4] (B_{Br} in orange) and the actual Fano factor of the particle current ($\text{Var}(I)/I$ in green) as functions of Σ/I, for terminals with $\beta_L = 2, \beta_R = 3, \mu_L = 1, \mu_R = 0$. The two graphs correspond to the transmission functions in the captions, with $\varepsilon_m = \operatorname{argmin}_\varepsilon \{(\delta_{\beta\mu} - \delta_\beta \varepsilon) g(\varepsilon)/\Delta f(\varepsilon)\}$ and $\mathcal{B}(a, b)$ the boxcar equal to 1 in the interval $[a, b]$.

Comparing these bounds with the boxcar bound, we can see roughly two regimes. One where $B_{\text{box}} \sim B \sim B_{\text{Br}}$ and one where $B_{\text{box}} \gg B \gg B_{\text{Br}}$. In the graphs found in figure 2 these correspond to the yellow/green regions and the blue/teal regions respectively.



(a) B_{Br}/B for $\beta_L = 1, \beta_R = 2, \mu_L = 0, \mu_R = 0$ (b) B/B_{box} for $\beta_L = 1, \beta_R = 2, \mu_L = 0, \mu_R = 0$



(c) B_{Br}/B for $\beta_L = 1, \beta_R = 5, \mu_L = 0, \mu_R = 1$ (d) B/B_{box} for $\beta_L = 1, \beta_R = 5, \mu_L = 0, \mu_R = 1$

Fig. 2: Comparison between our bound (B) and the bounds found in [4] (B_{Br}) and [5] (B_{box}) for the different possible regimes of particle (I) and energy (J) currents. The ratio B_{Br}/B is in the graphs to the left. The red regions in (a) and (c) denote when $B_{\text{Br}} > B$. The ratio B/B_{box} is in the graphs to the right.

IV. Conclusions

We show that it is possible to recover for arbitrary currents in thermoelectric transport (in the Landuaer-Buttiker regime) a TUR bound in the same vein as the classical TUR, as long as we give up on having a proportionality constant that is independent of the gradients. This illustrates that even in this case, keeping the signal to noise ratio of a current under control will require a steady production of entropy and hence steady dissipation of heat. In our opinion this highlights that TURs are a limitation on the design of computing devices on par with Landauer's principle. In the case of quantum computing, the need to shield the system from thermal fluctuations might lead to interesting challenges in this regard.

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